Journal of Statistical Physics, Vol. 49, Nos. 3/4, 1987

The Dynamics of Coupled Nonlinear Model Boltzmann Equations

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Received February 27, 1987; revision received June 19, 1987

A multispecies gas described by coupled nonlinear Boltzmann equations is studied as a dynamical system. Properties are determined of the N coupled nonlinear ODEs for the number densities obtained from the Boltzmann equations for the spatially uniform system of N species undergoing binary scattering, removal, and regeneration in the presence of an external force field and a reservoir of background gas. The physically realizable set Q, the nonnegative cone in the N-dimensional phase space of species number densities, is established as invariant under the flow. The fixed-point equations for the ODEs are shown to be equivalent to 2^N linear systems, and conditions for the stability and instability of the fixed points are then established. Stable fixed points are demonstrated to exist in Q by showing that they enter via a sequence of transcritical bifurcations as physical parameters are varied. For the two-species case the typical global structure of the solutions is established. Various particular cases are described including one which possesses an infinite family of periodic solutions and one that depends delicately upon initial conditions due to a separatrix that separates Q into two invariant sets.

KEY WORDS: Kinetic theory; dynamical systems; Boltzmann equations.

1. INTRODUCTION

Recently Boffi *et al.*[1] derived model equations describing the time evolution of spatially uniform, multispecies, rarefied gas mixtures with binary scattering, removal, and production processes in the presence of a reservoir of background gas and a conservative external field. The evolution equations, in the form of N coupled nonlinear ODEs for the

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0022-4715/87/1100-0607$05.00/0

number densities $\rho_i(t)$, $i = 1, 2, \ldots, N$, of the N reacting species were derived by integrating a set of coupled Boltzmann equations, namely

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \frac{1}{m_i}\mathbf{F}_i \cdot \frac{\partial}{\partial \mathbf{v}}\right) f_i(\mathbf{v}, t) \\ - \sum_{j=1}^{N+1} f_i(\mathbf{v}, t) \int_{\mathbb{R}^3} g_{ij} f_j(\mathbf{w}, t)\, d\mathbf{w} \\ + \sum_{j=1}^{N+1} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g_{ij}^s \Pi_{ij}^s(\mathbf{v}', \mathbf{w}' \to \mathbf{v})\, f_i(\mathbf{v}', t)\, f_j(\mathbf{w}', t)\, d\mathbf{v}'\, d\mathbf{w}' \\ + \sum_{j=1}^{N+1} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g_{ij,i}^c \chi_{ij}^c(\mathbf{v}', \mathbf{w}' \to \mathbf{v})\, f_i(\mathbf{v}', t)\, f_j(\mathbf{w}', t)\, d\mathbf{v}'\, d\mathbf{w}' \end{aligned} \tag{1}$$

over velocity with the assumption that all the interaction frequencies per unit density $g(|\mathbf{v}' - \mathbf{w}'|)$ are independent of velocity. The interaction processes were limited further by the assumption that species i is produced only in reactions between species i and species j and never in reactions between species j and k ($j \neq i$, $k \neq i$). Species $N+1$ is a background gas of constant density. With these restrictions the evolution equations for the number densities

$$\rho_i(t) = \int_{\mathbb{R}^3} f_i(\mathbf{v}, t)\, d^3v \tag{2}$$

become[1]

$$\dot{\rho}_i = \rho_i\left(-v_i - \sum_{j=1}^{N} C_{ij}\rho_j\right), \qquad i = 1, 2, \ldots, N \tag{3}$$

for all particle-conserving scattering kernels $\Pi_{ij}^s(\mathbf{v}', \mathbf{w}' \to \mathbf{v})$. The parameters v_i, which describes the reactions of species i with the background gas, and C_{ij}, which describes the interaction of species i with species j, are defined in Ref. 1. These $N + N^2$ parameters are restricted on physical grounds only by the following conditions[1]:

(i) $C_{ii} \geqslant 0$.

(ii) If $C_{ij} < 0$, then $C_{ji} \geqslant 0$.

Boffi *et al.* studied these equations for the cases $N = 2$ and $N = 3$ by numerical integration for various specific parameter values. In the present paper details of both local and global dynamics of the system are analyzed for various parameter regions and the bifurcations and resulting changes in the dynamics that occur as parameters are varied are investigated. Treating the evolution equations (3) as a nonlinear dynamical system, we view the solutions as trajectories or orbits in an appropriate N-dimensional state or phase space. Various theorems from dynamical systems theory, such as the Liapunov linearization theorem[2] (Liapunov's first method), and invariant manifold theorems, such as the center manifold theorem,[3] provide basic tools for the analysis.

After first establishing that the evolution equations are well posed for the nonnegative number densities, the fixed points (steady states or equilibria) of the system are examined for the N-species case and their stability properties are studied. The typical bifurcation that occurs as the parameters are varied in the N-species case is discussed. These results provide information on the steady states to which these N-species gas mixtures can evolve for various parameter values and on the qualitative differences in these steady states that result from different parameter values.

Specializing to the two-species case ($N=2$) allows some very strong results from the dynamical systems literature (e.g., the Poincaré–Bendixson theorem)[2,4] on two-dimensional vector fields to be used to establish global results for most of the physical parameter regions. Also, Peixoto's theorem[5] allows fairly strong statements to be made concerning the structural stability of the equations in this two-species case. These results provide complete knowledge of the long-term behavior of the two-species mixture. Two-species gas systems, as characterized by their parameter values, that show time-periodic variations in number density or great sensitivity to the initial preparation of the system are identified and discussed. Finally, the structural stability results for the two-species system guarantee that the introduction of additional interaction processes to the model will have no qualitative effect, provided the cross sections for these interactions are sufficiently small.

2. *N*-SPECIES GAS

2.1. Nonnegativity of Solutions

The interpretation of ρ_i, $i=1, 2,..., N$, as number densities requires that $\rho_i \geqslant 0$, so the appropriate state space on which to consider Eq. (3) is the set

$$Q=\{(\rho_1,..., \rho_N)\in \mathbb{R}^N \mid \rho_i \geqslant 0\} \tag{4}$$

which is an N-dimensional nonnegative cone. Given this physically appropriate space, which will be equipped with the relative topology,[6] it is necessary to know that Eqs. (3) determine a unique solution for every initial condition in Q and that the functions $\rho_i(t)$ so determined remain nonnegative. In other words, there must be a unique trajectory through each point in Q and the trajectory must be contained in Q. That Eqs. (3) determine unique solutions defined for some maximal time interval is automatic from standard existence, uniqueness, and extension results for ODEs.[2] The nonnegativity results from the following proposition.

Proposition 1. Q is invariant under the flow (evolution) defined on $\mathbb{R}^N$ by Eqs. (3).

Proof. Note that if the N-tuple $\rho(t) = (\rho_1(t),..., \rho_N(t))$ is any solution of Eqs. (3) and $\rho_k(T) = 0$ for some k and T (T positive, negative, or zero) in the domain of the solution, then $\rho_k(t) = 0$ for all t in that domain. This follows from the uniqueness of solutions with Cauchy data $\rho(T)$ and the fact that there exists a solution [verified by substitution of $\rho_k(t) = 0$ into Eqs. (3) and the standard existence theorem] with $\rho_k(t) \equiv 0$. Let the N-tuple $\rho(t) = (\rho_1(t),..., \rho_N(t))$ denote the solution of Eqs. (3) with initial condition $\rho(0)$ a point in Q. Suppose that there exists t' such that $\rho(t')$ is not in Q. Then there must exist k and T such that $\rho_k(0) > 0$, $\rho_k(t') < 0$, and (hence, by continuity) $\rho_k(T) = 0$. But then $\rho_k(t) \equiv 0$, contradicting $\rho_k(0) > 0$; hence, there exists no solution such that $\rho(0)$ is in Q and $\rho(t)$ is not in Q for some t. ∎

This proposition shows that it is permissible to restrict the system to the physically relevant set Q. The proof also shows that no number density can become zero in finite time. Finally, the key element in the proof applies to the case when $\rho_a(T) = \rho_b(T) = \cdots = 0$, thereby providing a corollary.

Corollary. Any set of the form

$$\{(\rho_1,..., \rho_N) \mid \rho_a = \rho_b = \cdots = 0\} \tag{5}$$

for any set of indices $\{a, b,...\}$ is an invariant manifold.

Proposition 1 and the standard extension theorem for ODEs also guarantee that either a solution is defined for all time (possibly diverging as $t \to \infty$) or else some number density goes to positive infinity in finite time.

This proposition shows that the evolution equations have the physically necessary property that the number densities remain nonnegative.

2.2. Fixed Points and Local Analysis

Fixed points of Eqs. (3), which represent steady states where removal and production processes balance for each species, are found by solving the system of N algebraic equations

$$0 = \bar{\rho}_i \left(-v_i - \sum_{j=1}^{N} C_{ij} \bar{\rho}_j \right) \tag{6}$$

for the N quantities $\bar{\rho}_i$, $i = 1, 2, \ldots, N$. Because each equation in this system is a product of two factors, this fixed-point problem can be solved using simple techniques from linear algebra. Let I be a (possibly empty) set of up to N integers and write card(I) for its cardinality. Also, denote by I' the set of integers between 1 and N that are not in I. Now let $\bar{\rho}_i = 0$ if $i \in I$ and let $\bar{\rho}_j$, $j \in I'$, satisfy the linear system

$$-v_j = \sum_{k \in I'} C_{jk} \bar{\rho}_k \tag{7}$$

Note that it is possible for a $\bar{\rho}_j$, $j \in I'$, to be zero or nonzero. The fixed-point problem (6) can then be solved completely by considering the 2^N possible index sets I and solving the linear system (7) for each of the index sets I'.

In order to characterize the structure of the set of fixed points, it is convenient to introduce some notation. Let $C_{I'}$ be the matrix obtained from $[C_{ij}]$ by eliminating each row and column indexed by an element of I, or equivalently by retaining those indexed by an element of I'. Similarly, let $v_{I'}$ and $\bar{\rho}_{I'}$ be the vectors obtained from $[v_i]$ and $[\bar{\rho}_i]$ by eliminating each entry indexed by an element of I. Using this notation, we can write the linear system (7) as

$$-v_{I'} = C_{I'} \bar{\rho}_{I'} \tag{8}$$

where $C_{I'}$ is a square matrix of dimension card$(I') = N - $card$(I)$, and all fixed points, solutions of Eq. (6), are of the form

$$\bar{\rho}_i = \begin{cases} 0, & i \in I \\ [\bar{\rho}_{I'}]_i, & i \in I' \end{cases} \tag{9}$$

The following proposition on the existence and nature of fixed points in $\mathbb{R}^N$ is then immediate from linear algebra.

Proposition 2.

1. $\bar{\rho}_i = 0$, $i = 1, 2, \ldots, N$, is a fixed point for all parameter values.
2. If, for a given I, $v_{I'} \neq 0$ and $\det[C_{I'}] \neq 0$, then Eq. (8) defines a single fixed point not equal to zero.
3. If, for a given I, $\det[C_{I'}] = 0$ and if $-v_{I'} \perp N(C_{I'}^*)$, then there is a d-dimensional manifold of fixed points of the form

$$\bar{\rho}_i = \begin{cases} 0, & i \in I \\ [\eta + p]_i, & i \in I' \end{cases}$$

where $\eta \in N(C_{I'})$, the null space of the matrix $C_{I'}$, d is the dimensionality of this null space, and p is any solution of Eq. (8). The condition relating $v_{I'}$ and the adjoint operator $C_{I'}^*$ guarantees that p exists. When $v_{I'} = 0$ the manifold passes through the origin.

This proposition does not guarantee that the fixed points are in Q, i.e., that all $\bar{\rho}_i(t) \geq 0$. Except for $(0,..., 0)$, there may be no fixed points in Q. However, because a nonsingular matrix such as appears in part 2 of the proposition is one to one and onto, it is possible to choose the elements of $v_{I'}$ so that $\bar{\rho}_{I'}$ contains only positive entries. Indeed, let $\{I'_k\}$ be a collection of pairwise disjoint index sets such that $\sum_k \operatorname{card}(I'_k) = N$. Then, by choosing the values of C_{ij}, $i, j = 1, 2,..., N$, such that $\det[C_{I'_k}] \neq 0$ for each k, it is possible to choose v_i, $i = 1, 2,..., N$, so that each element of $\bar{\rho}_{I'_k}$ is positive. Then, for each k, Q would contain an isolated fixed point with $\operatorname{card}(I'_k)$ positive number densities and $\operatorname{card}(I_k) = N - \operatorname{card}(I'_k)$ zero number densities. For example, let $k = 1, 2,..., N$ and $I_k = \{k\}$. Then, for $C_{ii} > 0$ and $v_i < 0$, $i = 1, 2,..., N$, there are N fixed points in Q of the form $\bar{\rho}_i = 0$, $i = 1, 2,..., N$, $i \neq k$, and $\bar{\rho}_k = -v_k/C_{kk}$. Even for parameter values satisfying this latter prescription, however, there may be more than $N + 1$ fixed points in Q. For example, for $N = 2$, $C_{11} = C_{22} = 1$, $C_{12} = C_{21} = 3$, and $v_1 = v_2 = -2$ there are $4 = 2^N$ fixed points in Q. In summary, there is always at least one fixed point in Q (the point $\bar{\rho}_k = 0$) and for any K, $1 \leq K \leq N + 1$, it is possible to choose parameters so that there are exactly K special fixed points in Q, at each of which a specified set of number densities is positive, provided that a number density that is positive at one of these special fixed points is zero at the other $K - 1$. At such a set of parameter values there may be other fixed points also in Q. It is reasonable to conjecture that for any N and any K, $1 \leq K \leq 2^N$, it is possible to find parameter values satisfying conditions (i) and (ii) of Section 1 and for which there are exactly K fixed points in Q. This conjecture is easily verified for $N = 1, 2$, and 3. But even if this conjecture is true in general, there are still other sets of parameter values, characterized by part 3 of Proposition 2, that satisfy conditions (i) and (ii) and for which there is an infinite number of fixed points in Q. An example of such a case for $N = 2$ is discussed in Section 3.6.

It is possible to make some statements concerning the stability of these various fixed points. The zero fixed point is the easiest to study and its stability properties will be described first.

Proposition 3.

1. If $v_i > 0$, $i = 1, 2,..., N$, then $(0,..., 0)$ is asymptotically stable.

2. If $v_i > 0$, $i = 1, 2, \ldots, N$, $i \neq k$, $v_k = 0$, and $C_{kk} > 0$, then $(0, \ldots, 0)$ is asymptotically stable.
3. If $v_i < 0$ for any i, then $(0, \ldots, 0)$ is unstable.

Proof. The linearized problem is governed by the Jacobian $J_{ij}(0, \ldots, 0) = v_i \delta_{ij}$, so if $v_i > 0$ for all i the linear problem is asymptotically stable, while if $v_i < 0$ for any i the linear problem is unstable. Parts 1 and 3 of the proposition then follow from standard linearization theorems.[2,7] If $v_k = 0$, the zero eigenvector of J_{ij} is $(0, \ldots, \rho_k, \ldots, 0)$ and then by the center manifold theorem[3] and the corollary to Proposition 1 the set $\{(\rho_1, \ldots, \rho_N) \in Q \mid \rho_i = 0 \ \forall i \neq k\}$ is a local center manifold for $(0, \ldots, 0)$. This manifold is asymptotically stable, since all eigenvalues of J_{ij} are negative except $-v_k = 0$ and the flow in the manifold is governed by $\dot{\rho}_k = -C_{kk} \rho_k \rho_k$. If $\rho_k(0) \geqslant 0$, then $\lim_{t \to \infty} \rho_k(t) = 0$; hence part 2 follows from the center manifold theorem. ∎

It is important to note that in case 2, zero is stable in Q not in $\mathbb{R}^N$; this is why Q is given the relative topology (Fig. 1). It also is possible to study a case with $v_j = 0$ for $j \in \{k_1, k_2, \ldots, k_M\}$ and $v_i > 0$ for $i \notin \{k_1, k_2, \ldots, k_M\}$. The center manifold theorem then allows the stability of $(0, \ldots, 0)$ to be deter-

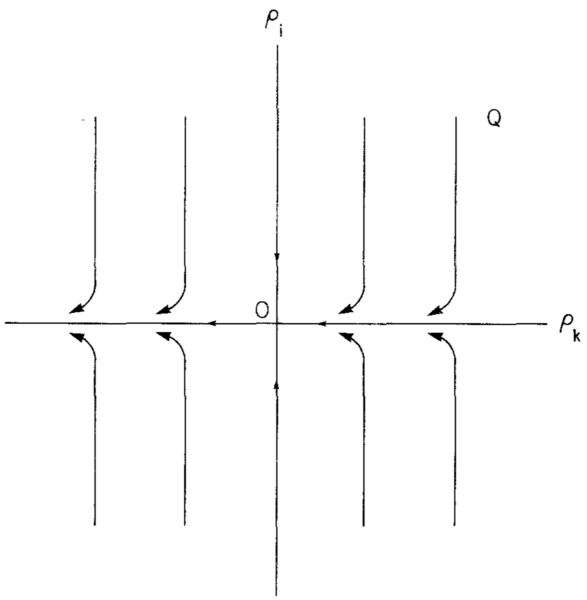

Fig. 1. The local flow near $0 = (0, \ldots, 0)$ when $v_k = 0$, $v_i > 0$, $i \neq k$, and $C_{kk} > 0$. In the physical state space Q, 0 is stable, while in $\mathbb{R}^N$ it is unstable.

mined by considering the flow on the M-dimensional center manifold, which is identical to the M-dimensional eigenspace of the zero eigenvalue (see corollary to Proposition 1). Thus, the flow is governed by the M ODEs obtained by fixing $\rho_i(t)=0, i \notin \{k_1, k_2,..., k_M\}$, in Eqs. (3). For example, if $C_{k_ik_j} \geq 0$ for every pair of k_ik_j the origin (0,..., 0) is easily shown to be asymptotically stable. If $v_1 = v_2 = 0$, the more complete results (not restricted to $C_{k_ik_j} \geq 0$) for the two-species equations derived in Section 3 can be applied to understand the flow in the two-dimensional center manifold of (0,..., 0).

As v_k passes through zero, a bifurcation occurs in which a fixed point $\bar{\rho}_k = -v_k/C_{kk}$, $\bar{\rho}_i = 0$, $i = 1, 2,..., N$, $i \neq k$, enters Q. In the invariant manifold defined by the ρ_k axis the flow is determined by

$$\dot{\rho}_k = \rho_k(-v_k - C_{kk}\rho_k) \tag{10}$$

which is the normal form for the transcritical bifurcation.[8] As v_i becomes negative, the new fixed point enters Q and becomes stable while (0,..., 0) becomes unstable. This stability result follows directly from the exchange of stability that occurs in a bifurcation at a simple eigenvalue.[9] (See Fig. 2.) The stability result also may be confirmed explicitly since

$$J_{ij}(0,..., -v_k/C_{kk},..., 0) = \begin{cases} (-v_i + C_{ik}v_k/C_{kk})\,\delta_{ij}, & i \neq k \\ C_{kj}v_k/C_{kk}, & i = k \end{cases} \tag{11}$$

has eigenvalues $-v_i + C_{ik}v_k/C_{kk}$, $i \neq k$, and v_k. Thus, if $C_{kk} > 0$, $v_i > 0$, $v_k < 0$, and $|v_k|$ is sufficiently small, all of these eigenvalues are negative and the new fixed point is stable by the linear stability theorems.[2,7]

To develop some stability results for the other fixed points, we note that the Jacobian matrix is given by

$$J_{ij}(\rho_1,..., \rho_N) = \delta_{ij}\Big(-v_i - \sum_{k=1}^{N} C_{ik}\rho_k\Big) - \rho_i C_{ij} \tag{12}$$

Thus, given an index set I,

$$J_{ij}(\bar{\rho}_1,..., \bar{\rho}_N) = \begin{cases} \left(-v_i - \sum_{k \in I'} C_{ik}\bar{\rho}_k\right)\delta_{ij}, & i \in I \\ -\bar{\rho}_i C_{ij}, & i \notin I \end{cases} \tag{13}$$

The spectral properties of this matrix can provide information on stability through the various linearization theorems.[2,7] The difficult task in determining these spectral properties is to find eigenvalues for a matrix of the form

$$J_{I'} = [-\bar{\rho}_i C_{ij}], \qquad i, j \in I' \tag{14}$$

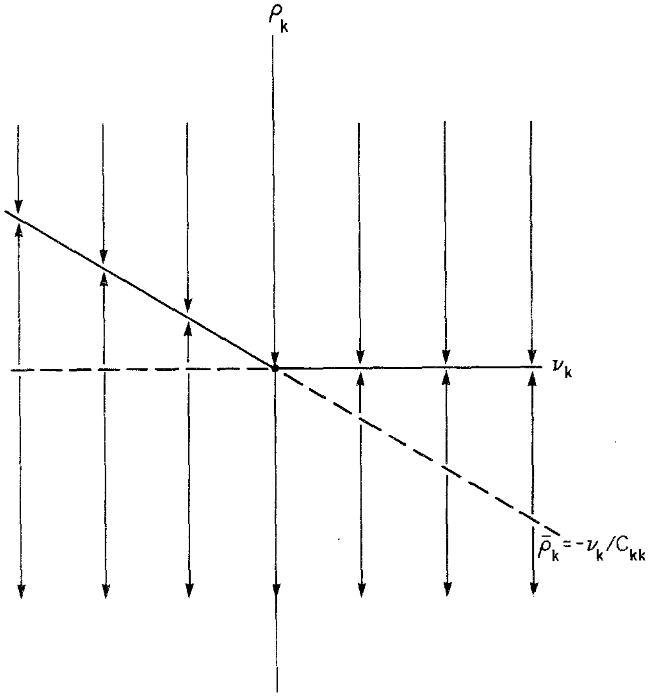

Fig. 2. The location of $\bar{\rho}_k$ versus ν_k and its stability as determined by the flow in the invariant manifold, the ρ_k axis.

Letting λ_i denote the eigenvalues of $J_{I'}$ (not necesarily distinct), one immediate observation is that

$$\sum_{i \in I'} \operatorname{Re}(\lambda_i) = -\sum_{i \in I'} \bar{\rho}_i C_{ii} \leqslant 0 \tag{15}$$

for every fixed point $(\bar{\rho}_1, \ldots, \bar{\rho}_N) \in Q$, since $C_{ii} \geqslant 0$. Thus, provided there is at least one i such that $C_{ii}\bar{\rho}_i \neq 0$, there is at least one eigenvalue with negative real part and therefore at least one contracting direction at every physically relevant fixed point.

A stronger result may be found by observing that

$$\det[J_{I'}] = (-1)^{N\text{-card}(I)} \det[C_{I'}] \prod_{i \in I'} \bar{\rho}_i \tag{16}$$

and that

$$\det[J_{I'}] = \prod_{i \in I'} \lambda_i \tag{17}$$

Now, defining

$$\Delta = \det[C_{I'}] \prod_{i \in I'} \bar{\rho}_i$$

it is easy to show the following result.

Proposition 4.

1. If $\Delta < 0$, there is a real, positive eigenvalue.
2. If $\Delta = 0$, there is a zero eigenvalue.
3. All eigenvalues have negative real part only if $\Delta > 0$.

Proof. Part 2 is obvious and part 3 follows from parts 1 and 2. To prove part 1, suppose $\Delta < 0$. First, let $N - \text{card}(I)$, the dimension of $J_{I'}$, be even. Then $\prod \lambda_i = \Delta < 0$. If $\text{Im}(\lambda_i) \neq 0$, then $\bar{\lambda}_i$ also is an eigenvalue, and $\lambda_i \bar{\lambda}_i > 0$. Thus, there must be an even number (not zero) of eigenvalues with $\text{Im}(\lambda_i) = 0$ and an odd number of these must be negative. Hence, there must be at least one real, positive eigenvalue. Now, let N-card(I) be odd. Then $\prod \lambda_i = -\Delta > 0$. Hence, there is an odd number of λ_i with $\text{Im}(\lambda_i) = 0$ and an even number of these must be negative. Hence, again there must be at least one real, positive eigenvalue. ∎

Some stability results follow from the above. Let I be an index set, and suppose that a corresponding fixed point is physical, i.e., $\bar{\rho}_i \geqslant 0$, $i = 1, 2,..., N$. Clearly, from Eq. (13) this fixed point is unstable if $(-v_i - \sum_{j \in I'} C_{ij} \bar{\rho}_j) > 0$ for some $i \in I$. Also, from Proposition 4, a fixed point in Q is unstable if, for the index set I with $\bar{\rho}_i > 0$ for all $i \in I'$, the condition $\det[C_{I'}] < 0$ holds. This latter condition is useful since it requires no knowledge of the Jacobian and only lower bounds on the $\bar{\rho}_i$. Also, it is possible to state that the Jacobian will have a zero eigenvalue if either (1) $\bar{\rho}_i = 0$ for any $i \in I'$, (2) $\det[C_{I'}] = 0$, or (3) $(-v_i - \sum_{j \in I'} C_{ij} \bar{\rho}_j) = 0$ for any $i \in I$. (Recall that, if $i \in I$, then $\bar{\rho}_i = 0$; however, one or more $\bar{\rho}_i$, $i \notin I$, which satisfy a linear system also could be zero.) Fixed points such as these, with singular Jacobians, are interesting for two reasons: first, linearization theorems break down at such points, requiring more subtle stability analyses, and second, in view of the implicit function theorem, such a fixed point may bifurcate into several fixed points at nearby parameter values. In fact, as will be shown below, conditions 1 and 3 will lead to the typical bifurcation wherein a new fixed point enters Q.

The zero-eigenvalue condition, condition 2, and the existence of manifolds of fixed points (Proposition 2, part 3) require a very specific condition on the parameters, namely that some determinant of C_{ij} be zero. However, it is always possible to select new values of the C_{ij} that are as close as desired to the original values, but for which $\det[C_{I'}] \neq 0$ for every index set I. This results from the following proposition.

Proposition 5. Define the good sets

$$GS_{I'} = \{C_{ij} \in \mathbb{R}^{N^2} \mid \det[C_{I'}] \neq 0\}$$

and the physical set

$$P = \{C_{ij} \in \mathbb{R}^{N^2} \mid C_{ii} \geqslant 0, \text{ if } C_{ij} < 0 \text{ then } C_{ji} \geqslant 0\}$$

Then $\{\bigcap_{I'} GS_{I'}\} \cap P$ is open and dense in P.

Proof. Each $GS_{I'}$ is obviously open. Now let $\lambda_{I'} = \min\{|\mathrm{Re}\,\lambda| \mid \lambda$ an eigenvalue of $C_{I'}$ with $|\mathrm{Re}\,\lambda| \neq 0\}$. Then there exists $\varepsilon > 0$ such that $\varepsilon < \min\{\lambda_{I'} \mid I'$ an index set$\}$. Then $\det[C_{I'} - \delta] \neq 0$ for any I' and any δ, $0 < \delta < \varepsilon$. Thus, $GS = \bigcap_{I'} GS_{I'}$ is open and dense in $\mathbb{R}^{2^N}$ and so the proposition follows. ∎

This proposition guarantees that whenever the parameters C_{ij} are chosen so that a manifold of fixed points exists, then it is only necessary to change the parameters C_{ii} by any small amount to destroy the manifold. Thus, *typically* the system could have anywhere between one and 2^N isolated fixed points in Q, but no manifolds of fixed points. Such manifolds are exceptional. Most importantly, excluding the exceptional parameter values, there can be at most one fixed point with all positive number densities. If there is such a fixed point and $\det[C_{ij}] < 0$, then it is unstable by the first instability criterion. Further, since the typical dense parameter set has $C_{ii} > 0$, $i = 1, 2,..., N$, by Eq. (15) such a fixed point typically would be a saddle point.

With the parameters restricted to the good set described in Proposition 5 the typical bifurcation that occurs as a fixed point moves into Q can be described. Let $\bar{\rho}$ be a fixed point of the form $(\bar{\rho}_1,..., \bar{\rho}_n, 0,..., 0)$ with $\bar{\rho}_i > 0$, $i = 1, 2,..., n$. Then $\bar{\rho}$ is determined by the linear system

$$-\nu_i = \sum_{j=1}^{n} C_{ij}\bar{\rho}_j, \qquad i = 1, 2,..., n \tag{18}$$

Also, let $\tilde{\rho} = (\tilde{\rho}_1,..., \tilde{\rho}_{n+1}, 0,..., 0)$ be the fixed point determined by

$$-\nu_i = \sum_{j=1}^{n+1} C_{ij}\tilde{\rho}_j, \qquad i = 1, 2,..., n+1 \tag{19}$$

Denote by $[C_{ij}]_n$ and $[C_{ij}]_{n+1}$ the square matrices in these linear systems (of course, $[C_{ij}]_{n+1}$ contains $[C_{ij}]_n$), and note that on the good set of parameter values $\det[C_{ij}]_n \neq 0$ and $\det[C_{ij}]_{n+1} \neq 0$.

A very useful relation between $(\bar{\rho}_1,..., \bar{\rho}_n, 0,..., 0)$ and $(\tilde{\rho}_1,..., \tilde{\rho}_{n+1}, 0,..., 0)$ is given by

$$-v_{n+1} - \sum_{j=1}^{n} C_{n+1,j}\bar{\rho}_j = \frac{\det[C_{ij}]_{n+1}}{\det[C_{ij}]_n}\tilde{\rho}_{n+1} \tag{20}$$

To derive this, let $\mathrm{Cof}_{ij}^{(n+1)}$ be the cofactor of C_{ij} in $[C_{ij}]_{n+1}$ and simularly let $\mathrm{Cof}_{ij}^{(n)}$ denote the cofactor of C_{ij} in $[C_{ij}]_n$. Here the cofactor of the element C_{ij} in an n-dimensional square matrix C is $(-1)^{i+j}$ times the determinant of the $(n-1)$-dimensional cofactor matrix produced by eliminating row i and column j from C. Then, using the expression for the inverse of a matrix, i.e., Cramer's rule,

$$\tilde{\rho}_{n+1} = \frac{1}{\det[C_{ij}]_{n+1}} \sum_{i=1}^{n+1} -v_i \,\mathrm{Cof}_{i,n+1}^{(n+1)} \tag{21}$$

But by expanding the cofactors using the $(n+1)$th row of $[C_{ij}]_{n+1}$, which is only the nth row of the cofactor matrices obtained by eliminating row i and column $n+1$ of $[C_{ij}]_{n+1}$, it can be shown that

$$\mathrm{Cof}_{i,n+1}^{(n+1)} = -\sum_{j=1}^{n} C_{n+1,j}\,\mathrm{Cof}_{ij}^{(n)}, \qquad i < n+1 \tag{22}$$

and also

$$\mathrm{Cof}_{n+1,n+1}^{(n+1)} = \det[C_{ij}]_n \tag{23}$$

Thus

$$\tilde{\rho}_{n+1} = \frac{1}{\det[C_{ij}]_{n+1}} \left\{ -\sum_{i=1}^{n}\sum_{j=1}^{n} -v_i C_{n+1,j}\,\mathrm{Cof}_{ij}^{(n)} - v_{n+1}\det[C_{ij}]_n \right\} \tag{24}$$

or

$$\frac{\det[C_{ij}]_{n+1}}{\det[C_{ij}]_n}\tilde{\rho}_{n+1} = -v_{n+1} - \sum_{j=1}^{n} C_{n+1,j} \sum_{i=1}^{n} -v_i \frac{\mathrm{Cof}_{ij}^{(n)}}{\det[C_{ij}]_n} \tag{25}$$

and since

$$\bar{\rho}_j = \sum_{i=1}^{n} -v_i \frac{\mathrm{Cof}_{ij}^{(n)}}{\det[C_{ij}]_n} \tag{26}$$

this establishes the relationship given by Eq. (20).

Equation (20) guarantees that there are parameter values at which $\tilde{\rho}_{n+1}=0$. Any set of parameter values that satisfy

$$0=-v_{n+1}-\sum_{j=1}^{n} C_{n+1,j}\bar{\rho}_j \tag{27}$$

where $\bar{\rho}_j$ satisfy Eq. (18), $\det[C_{ij}]_n\neq 0$, and $\det[C_{ij}]_{n+1}\neq 0$ will do. Such a set of parameters values [i.e., a set for which Eq. (27) is satisfied] will be called critical. For example, for any good values of C_{ij}, $i, j=1, 2,..., n+1$, and v_i, $i=1, 2,..., n$, choose v_{n+1} to satisfy (27). But if $\tilde{\rho}_{n+1}=0$, then Eq. (19) becomes

$$-v_i=\sum_{j=1}^{n} C_{ij}\tilde{\rho}_j \tag{28}$$

while $\det[C_{ij}]_n\neq 0$ and $\det[C_{ij}]_{n+1}\neq 0$ imply that $\bar{\rho}$ and $\tilde{\rho}$ are unique, so $\bar{\rho}=\tilde{\rho}$ in this case. It is therefore possible (e.g., by varying v_{n+1}) to cause $\tilde{\rho}_{n+1}$ to vary from negative to positive while the fixed point $\tilde{\rho}$ moves through the fixed point $\bar{\rho}$. By continuity, near any such critical parameter values it must be that $\tilde{\rho}_i>0$, $i=1, 2,..., n$, since this holds for $\bar{\rho}$. A bifurcation results as the fixed point ρ moves into or out of Q.

From Eq. (13) the Jacobian matrix at $\bar{\rho}$ is

$$J_{ij}(\bar{\rho}_i,..., \bar{\rho}_n, 0,..., 0)=\begin{cases} -\bar{\rho}_i C_{ij}, & i=1, 2,..., n \\ \left(-v_{n+1}-\sum_{j=1}^{n} C_{n+1,j}\bar{\rho}_j\right)\delta_{n+1,j}, & i=n+1 \\ \left(-v_i-\sum_{j=1}^{n} C_{ij}\bar{\rho}_j\right)\delta_{ij}, & i=n+2,..., N \end{cases} \tag{29}$$

Thus, in light of (20),

$$J_{ij}(\bar{\rho}_1,..., \bar{\rho}_n, 0,..., 0)=\begin{cases} -\bar{\rho}_i C_{ij}, & i=1, 2,..., n \\ s\tilde{\rho}_{n+1}\,\delta_{n+1,j}, & i=n+1 \\ \left(-v_i-\sum_{j=1}^{n} C_{ij}\bar{\rho}_j\right)\delta_{ij}, & i=n+2,..., N \end{cases} \tag{30}$$

where

$$s=\det[C_{ij}]_n/\det[C_{ij}]_{n+1} \tag{31}$$

This Jacobian has an eigenvalue

$$\lambda=s\tilde{\rho}_{n+1} \tag{32}$$

Clearly, this eigenvalue passes through zero as $\tilde{\rho}_{n+1}$ does. At the critical parameter values $\tilde{\rho}_{n+1}=0$ and $\lambda=0$, the zero eigenvector of $J_{ij}(\bar{\rho})$ is of the form $(\gamma_1,..., \gamma_n, 1, 0,..., 0)$, where

$$\sum_{j=1}^{n} C_{ij}\gamma_j = -C_{i,n+1} \tag{33}$$

This has a unique solution γ_i, $i=1, 2,..., n$, since $\det[C_{ij}]_n \neq 0$. Thus, at the critical parameter values where $\bar{\rho}=\tilde{\rho}$ there is a center manifold through $\tilde{\rho}$ and tangent to $(\gamma_1,..., \gamma_n, 1, 0,..., 0)$. If, when $\lambda=s\tilde{\rho}_{n+1}=0$, the Jacobian at $\tilde{\rho}=\bar{\rho}$ has all eigenvalues except λ with negative real part, then the bifurcation is a transcritical bifurcation. This is a bifurcation at a simple eigenvalue and must exhibit an exchange of stability.[9] The $\tilde{\rho}_{n+1}$ axis can then be used to provide local coordinates in which the bifurcation appears as in Fig. 3.

The fixed point $\bar{\rho}$ is then stable when $\lambda=s\tilde{\rho}_{n+1}<0$ and the exchange of stability requires that $\tilde{\rho}$ be unstable when $\lambda<0$, while for $\lambda>0$, $\bar{\rho}$ is unstable and $\tilde{\rho}$ is stable. This suggests the following result.

Proposition 6. Let N and K be given, $N>0$, $0 \leqslant K \leqslant N$. Then there exists a set of parameter values in $\bigcap_{I'} GS_{I'} \cap P$ such that the N-species equations (3) possess a stable fixed point with K positive number densities and $N-K$ zero number densities.

Proof. Let the fixed point be of the form $(\tilde{\rho}_1, \tilde{\rho}_2,..., \tilde{\rho}_k, 0,..., 0)$ with $\tilde{\rho}_i>0$. If $K=0$, choose $v_i>0$, $i=1, 2,..., N$, and then this proposition

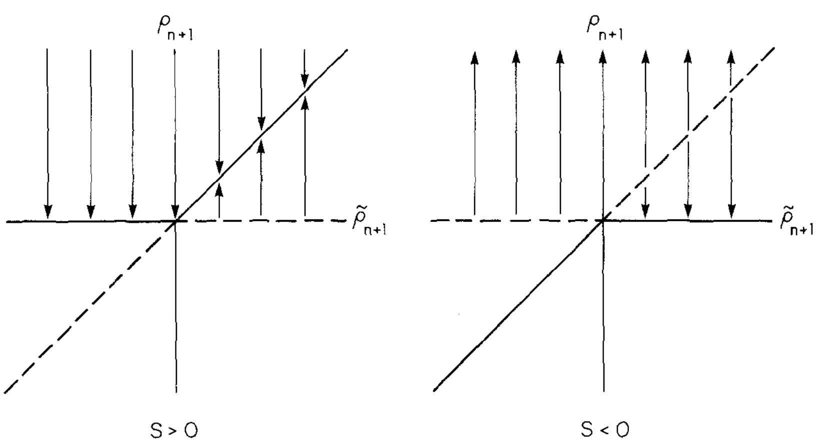

Fig. 3. The local flow in the invariant manifold, the ρ_{n+1} axis. The two exchanges of stability possibilities ($s>0$, $s<0$) are shown with the flow in Q.

follows from Proposition 3. If $K=1$, choose $C_{11}>0$, $v_1<0$, $v_i>0$, $i=2,\ldots,N$; then $(-v_1/C_{11}, 0,\ldots, 0)$ is a stable fixed point. The proof is now completed by induction. Suppose the proposition is true for $K=n$ and let $(\bar{\rho}_1,\ldots, \bar{\rho}_n, 0, 0,\ldots, 0)$ be the stable fixed point. By Proposition 4, $\det[C_{ij}]_n>0$. Now, because of the transcritical bifurcation, it is possible to choose ε, $v_{n+1}, C_{i,n+1}$, and $C_{n+1,i}$, $i=1, 2,\ldots, n+1$, so that $\det[C_{ij}]_{n+1}>0$, $\varepsilon> -v_{n+1}-\sum_{j=1}^{n} C_{n+1}\bar{\rho}_j>0$, and so that $(\tilde{\rho}_1,\ldots, \tilde{\rho}_{n+1}, 0,\ldots, 0)$ is stable. It is clear that the parameters so chosen can be in $\bigcap_{I'} GS_{I'} \cap P$ (e.g., choose $C_{n+1,n+1}>0$ and $C_{i,n+1}$, $C_{n+1,i}$, $i=1, 2,\ldots, n$, small in magnitude). Thus, the proposition holds for any K. ∎

More graphically, the fixed point $(-v_1/C_{11}, 0,\ldots, 0)$ is brought into Q in a transcritical bifurcation at $(0,\ldots, 0)$. It becomes stable and $(0,\ldots, 0)$ becomes unstable. The point $(-v_1/C_{11}, 0,\ldots, 0)$ is guaranteed to be stable for some range $\varepsilon_1> -v_1>0$ and $C_{11}>0$. Now fix v_1 in this range and fix C_{11} and vary v_2, C_{22}, C_{12}, and C_{21} to cause a transcritical bifurcation at $(-v_1/C_{11}, 0,\ldots, 0)$. Again the exchange of stability takes place and a new fixed point of the form $(\bar{\rho}_1^{(2)}, \bar{\rho}_2^{(2)}, 0,\ldots, 0)$ becomes stable as it enters Q and remains stable for some range of parameters with $\varepsilon_2> -v_2-C_{21}(-v_1/C_{11})>0$ and $\det[C_{ij}]_2=C_{11}C_{22}-C_{12}C_{21}>0$. This process is continued up to a stable fixed point of the form $(\bar{\rho}_1^{(K)},\ldots, \bar{\rho}_K^{(K)}, 0,\ldots, 0)$ (Fig. 4).

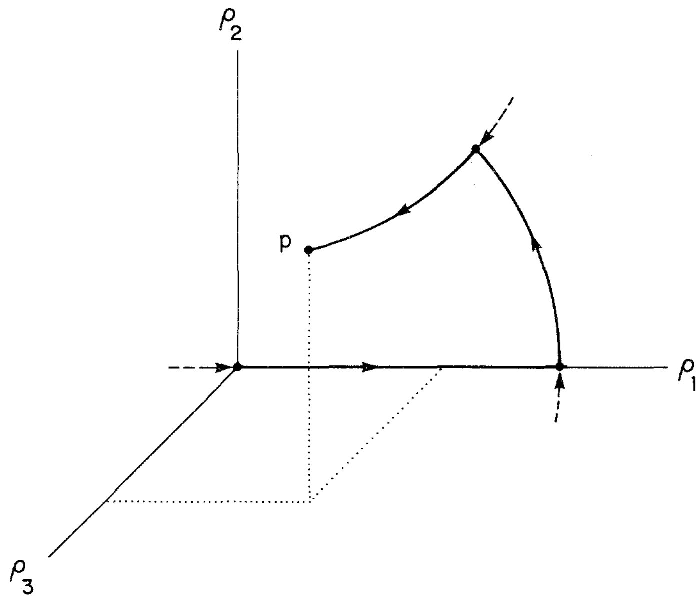

Fig. 4. A stable fixed point created by a sequence of three transcritical bifurcations.

It is easy to write down the necessary conditions for the existence of the stable fixed point created by such a sequence of K transcritical bifurcations:

1a. $C_{11} > 0$

1b. $-v_1 > 0$

2a. $C_{11}C_{22} - C_{12}C_{21} > 0$

2b. $-v_2 - C_{21}(-v_1/C_{11}) > 0$

3a. $\det[C_{ij}]_3 > 0$

$\vdots$

ma. $\det[C_{ij}]_m > 0$

mb. $-v_m - \sum_{J=1}^{m-1} C_{mj}\bar{\rho}_j^{(m-1)} > 0$

$\vdots$

Ka. $\det[c_{ij}]_K > 0$

Kb. $-v_K - \sum_{j=1}^{K} C_{Kj}\bar{\rho}_j^{(K-1)} > 0$

and if $K < N$

$$-v_i - \sum_{j=1}^{K} C_{ij}\bar{\rho}_j^{(K)} < 0, \qquad i = K+1,..., N$$

where $\rho_i^{(m-1)}$ satisfy

$$-v_i = \sum_{j=1}^{m-1} C_{ij}\bar{\rho}_j^{(m-1)} \tag{34}$$

and $\det[C_{ij}]_m$ is the determinant of the $m \times m$ matrix with $i, j = 1, 2,..., m$. Clearly, the labeling of indices is arbitrary.

2.3. Some Three-Species Examples

In this subsection the three specific three-species systems studied numerically by Boffi *et al.*[1] (Table I) will be examined briefly. The first

Table I. Parameter Values Used by Boffi *et al.*[1]

Case	v_1	C_{11}	C_{12}	C_{13}	v_2	C_{21}	C_{22}	C_{23}	v_3	C_{31}	C_{32}	C_{33}
1	0	1	1	1	0	2	2	2	0	3	3	3
2	−3	7	2	0	1	−4	0	2	.5	0	−1	0
3	−3	1	2	0	1	−4	0	2	.5	0.	−1	0

case is quite trivial; $(0, 0, 0)$ is the only fixed point in Q and it is easy to prove that every solution with initial condition in Q evolves to this fixed point. There is a manifold of fixed points, but it intersects Q only at $(0, 0, 0)$. This physical fixed point is thus a double zero: the system is at a bifurcation point. For instance, the introduction of any linear loss mechanism, i.e., a $v_i > 0$, no matter how small, will produce a qualitatively different flow.

The second case is much more interesting. Now physically relevant fixed points are located at $(0, 0, 0)$, $(3/7, 0, 0)$, $(1/4, 5/4, 0)$, and $(2/7, 1/2, 1/14)$. This series of points corresponds to the discussion at the end of the previous section. From that section the conditions to be satisfied (1a)–(3b) become

1a. $C_{11} = 7 > 0$

1b. $-v_1 = 3 > 0$

2a. $C_{11}C_{22} - C_{12}C_{21} = 8 > 0$

2b. $-v_2 - C_{21}(-v_1/C_{11}) = 5/7 > 0$

3a. $\det[C_{ij}]_3 = 14 > 0$

3b. $-v_3 - C_{31}\bar{\rho}_1^{(2)} - C_{32}\bar{\rho}_2^{(2)} = 3/4 > 0$

These four fixed points can be viewed as having been born in a sequence of transcritical bifurcations as a result of a specific series of parameter variations. Even without explicit knowledge of the eigenvalues of the Jacobian at $(2/7, 1/2, 1/14)$, the appearance of the four fixed points satisfying conditions 1a–3b suggests that $(2/7, 1/2, 1/4)$ is stable. In fact, since no additional static bifurcations occur, it will be stable unless a dynamic bifurcation occurs for this series of parameter variations. Also, conditions 1b, 2b, and 3b do mean that $(0, 0, 0)$, $(3/7, 0, 0)$, and $(1/4, 5/4, 0)$ are unstable. The numerical results of Boffi *et al.*[1] are consistent with the stability of $(2/7, 1/2, 1/14)$, and, in fact, the eigenvalues of the Jacobian there are $-0.1534...$ and $-0.4100... \pm i(2.2801...)$; hence, $(2/7, 2, 1/14)$ is stable.

In case 3 of Table I there are again four fixed points in Q: $(0, 0, 0)$, $(3, 0, 0)$, $(1/4, 11/8, 0)$, and $(2, 1/2, 7/2)$. Again conditions 1a–3b are satisfied, so $(0, 0, 0)$, $(3, 0, 0)$, and $(1/4, 11/8, 0)$ are unstable and $(2, 1/2, 7/2)$ can be viewed as the end result of a sequence of transcritical bifurcations. This suggests that $(2, 1/2, 7/2)$ is stable, which is again consistent with the earlier numerical results.[1] In fact, the eigenvalues of the Jacobian at $(2, 1/2, 7/2)$ are $-0.3223...$ and $-0.08880... \pm i(4.6834...)$, and so $(2, 1/2, 7/2)$ is stable.

2.4. Discussion

It should be clear from these results that Proposition 6 is useful as a statement concerning the types of equilibria that a gas system described by Eqs. (3) can possess, but not as a method to determine behavior in specific cases. On the other hand, the stable equilibria in the interior of Q whose presence is established by Proposition 6 are hyperbolic[7,8] and therefore a real gas system dominated by interactions like those explicitly modeled by Eqs. (3) will possess stable equilibria in the presence of sufficiently small interactions not included in Eqs. (3). It also follows from Proposition 2 that the only static bifurcations that can occur consist of transcritical bifurcations (possibly many simultaneously) and bifurcations involving manifolds of fixed points ("critical" transcritical bifurcations). This implies that a stable equilibrium in the interior of Q could lose its stability by either passing through a manifold of fixed points or in a dynamic bifurcation. It is therefore likely that either the conditions listed after Proposition 6 are both necessary and sufficient for the gas system to have a stable equilibrium point or else the number densities can undergo a persistent time-dependent behavior (e.g., a periodic solution in this parameter range).

Another property of such a gas system concerns the stabilizing effect of self-removal. In a one-species ($N=1$) system with production from the background ($v_1<0$) the existence of self-removal ($C_{11}>0$) guarantees the existence of a stable fixed point $\bar{\rho}_1=-C_{11}/v_1>0$. That is, the self-removal stabilizes the system. For the general case ($N>1$) a similar phenomenon occurs. The simple observation, based on Eq. (15), is that associated with a fixed point where species i survives ($\bar{\rho}_i>0$) and also undergoes self-removal there is a stable manifold, a special set of initial number densities that evolve to the fixed point, even if it is unstable. Again, this property will survive the introduction of additional interaction processes, provided they are sufficiently small.

3. TWO-SPECIES GAS

For the two-species equations

$$\dot{\rho}_1=\rho_1(-v_1-C_{11}\rho_1-C_{12}\rho_2) \tag{35}$$

$$\dot{\rho}_2=\rho_2(-v_2-C_{21}\rho_1-C_{22}\rho_2) \tag{36}$$

Table II lists all the isolated fixed points and the appropriate stability-governing eigenvalues at each. For most parameter values this information essentially determines the local dynamics. However, the two species

Table II. Fixed Points and Eigenvalues for $N=2$

$\bar{\rho}_1$	$\bar{\rho}_2$	Eigenvalues
0	0	$-\nu_1,\ -\nu_2$
0	$-\nu_2/C_{22}$	$-\nu_1+C_{12}(\nu_2/C_{22}),\ +\nu_2$
$-\nu_1/C_{11}$	0	$-\nu_2+C_{21}(\nu_1/C_{11}),\ +\nu_1$
$\dfrac{C_{12}\nu_2-\nu_1C_{22}}{C_{11}C_{22}-C_{12}C_{21}}$	$\dfrac{C_{21}\nu_1-\nu_2C_{11}}{C_{11}C_{22}-C_{12}C_{21}}$	$-\frac{1}{2}(C_{11}\bar{\rho}_1+C_{22}\bar{\rho}_2)$ $\pm\frac{1}{2}[(C_{11}\bar{\rho}_1+C_{22}\bar{\rho}_2)+4C_{12}\bar{\rho}_1\bar{\rho}_2]^{1/2}$

equations are sufficiently special that a number of global results may be established. These results concern the ω-limit (omega-limit) sets of points in Q. Let $\rho(t;p)=(\rho_1(t;p),\ \rho_2(t;p))$ be the unique solution of Eqs. (36) with $\rho(0;p)=p\in Q$. If $\rho(t;p)$ is defined for all $t\geqslant 0$, the ω-limit of p, denoted $\omega(p)$, is the set $\omega(p)=\{q\in Q|$ there exists $\{t_n\}$, an increasing sequence unbounded above s.t. $q=\lim_{n\to\infty}\rho(t_n;p)\}$. The ω-limit of p is where the trajectory through p accumulates and contains the asymptotic fate of $\rho(t;p)$ as $t\to\infty$. The ω-limit sets are clearly of great interest in the gas evolution problem since they specify the eventual long-time behavior of the system.

For two-dimensional systems there are very strong results which classify the form of ω-limit sets in compact positive invariant sets: the Poincaré–Bendixson theorem and its corollaries.[2,4,8] A set $K\subset\mathbb{R}^2$ is positive invariant if $\rho(t;p)\in K$ for all $t\geqslant 0$ and all $p\in K$. The Poincaré–Bendixson theorem states that if the system has only a finite number of fixed points in K, then the ω-limit of any $p\in k$ takes one of three forms:

(i) A single fixed point.

(ii) A closed orbit (e.g., a limit cycle).

(iii) A finite number of fixed points and orbits connecting them.

For the two-species equations (36), it is possible for most parameter regimes to eliminate cases (ii) and (iii) and establish that for almost all initial conditions $p\in Q$, $\rho(t;p)$ tends to one particular fixed point as $t\to\infty$.

3.1. Compact Positive Invariant Sets

In this subsection the existence of positive invariant sets in the two-dimensional nonnegative cone (quadrant) Q is established via the somewhat tedious proof of the following proposition.

Proposition 7. Let $p\in Q$. If $C_{11}>0$ and $C_{22}>0$, then there exists a compact, positive invariant set K of the form $\{(\rho_1, \rho_2) | 0 \leqslant \rho_1 \leqslant k_1, 0 \leqslant \rho_2 \leqslant k_2\}$ for the flow of Eqs. (36) with $p \in K$.

Expressed differently, this proposition asserts that given any initial condition p in Q, there exists a compact (i.e., closed and bounded) set K such that the solution $\rho(t; p)$ of Eqs. (36) is defined for all $t > 0$ and such that $\rho(t; p) \in K$ for all $t \geqslant 0$, provided only that C_{11} and C_{22} are greater than zero. Thus, as long as there is any self-removal at all, no matter how small or large, the number densities remain uniformly bounded (and by Proposition 1, nonnegative) for all positive time.

Proof of Proposition 7 is contained in the following five lemmas and three corollaries, each of which establishes, for different parameter regimes, the existence of a compact set K containing p and such that no positive orbit can exit K. The existence of such a compact set ensures that $\rho(t; p)$ is defined for all $t \geqslant 0$ (see Ref. 2) and thus that K is positive invariant.

Write $p = (r_1, r_2)$ with $r_1 \geqslant 0$ and $r_2 \geqslant 0$. Each K will be a box of the form

$$K = \{(\rho_1, \rho_2) | 0 \leqslant \rho_1 \leqslant k_1, 0 \leqslant \rho_2 \leqslant k_2\} \tag{37}$$

with $r_1 < k_1$, and $r_2 < k_2$. Each lemma and corollary will assert the existence of k_1 and k_2 such that $\dot{\rho}_1 < 0$ on the line $\rho_1 = k_1$ and $\dot{\rho}_2 < 0$ on the line $\rho_2 = k_2$ (Fig. 5). The positive invariance of K then follows from this and Proposition 1.

Lemma 1. Let $C_{11} > 0$, $C_{22} > 0$, $C_{12} \geqslant 0$, $C_{21} \geqslant 0$, $\nu_1 \geqslant 0$, $\nu_2 \geqslant 0$. Then k_1 and k_2 exist.

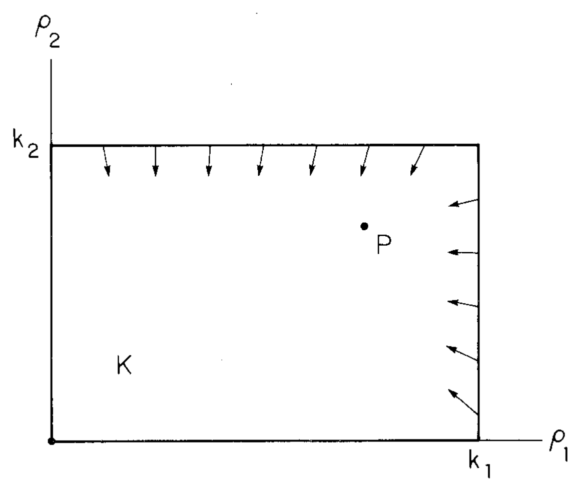

Fig. 5. A compact, positive invariant box K inside the noncompact but invariant set Q.

Proof. Let $k_1 > r_1$ and $k_2 > r_2$. Then, since $\dot{\rho}_1 < 0$ and $\dot{\rho}_2 < 0$ in all of Q, the lemma follows. ∎

Lemma 2. Let $C_{11} > 0$, $C_{22} > 0$, $C_{12} \geqslant 0$, $C_{21} \geqslant 0$, $v_1 \leqslant 0$, $v_2 \leqslant 0$. Then k_1 and k_2 exist.

Proof. Let $k_1 > \max\{r_1, -v_1/C_{11}\}$ and $k_2 > \max\{r_2, -v_2/C_{22}\}$. If $\rho_1 = k_1$, then

$$-C_{11}\rho_1 = -C_{11}k_1 < -C_{11}(-v_1/C_{11}) = v_1$$

Hence

$$-v_1 - C_{11}\rho_1 - C_{12}\rho_2 \leqslant -v_1 - C_{11}\rho_1 < 0$$

Thus $\dot{\rho}_1 < 0$. Similarly, if $\rho_2 = k_2$, $\dot{\rho}_2 < 0$. ∎

Lemma 3. Let $C_{11} > 0$, $C_{22} > 0$, $C_{12} \geqslant 0$, $C_{21} < 0$, $v_1 \geqslant 0$, $v_2 \geqslant 0$. Then k_1 and k_2 exist.

Proof. Let

$$k_1 > r_1, \qquad k_2 > \max\left\{r_2, \frac{-C_{21}}{C_{22}}k_1 - \frac{v_2}{C_{22}}\right\}$$

Then, if $\rho_1 = k_1$, $-v_1 - C_{11}\rho_1 - C_{12}\rho_2 < 0$, so $\dot{\rho}_1 < 0$. If $\rho_2 = k_2$ and $0 \leqslant \rho_1 \leqslant k_1$, then $-C_{21}\rho_1 \leqslant -C_{21}k_1$, so

$$-v_2 - C_{21}\rho_1 - C_{22}\rho_2 \leqslant -v_2 - C_{21}k_1 - C_{22}k_2$$

But $-C_{22}k_2 < C_{21}k_1 + v_2$, so $-v_2 - C_{21}\rho_1 - C_{22}\rho_2 < 0$ and thus $\dot{\rho}_2 < 0$. ∎

Corollary. Let $C_{11} > 0$, $C_{22} > 0$, $C_{12} < 0$, $C_{21} \geqslant 0$, $v_1 \geqslant 0$, $v_2 \geqslant 0$. Then k_1 and k_2 exist.

Proof. Permute the indices of Lemma 3. ∎

Lemma 4. Let $C_{11} > 0$, $C_{22} > 0$, $C_{12} \geqslant 0$, $v_1 \leqslant 0$, $v_2 \geqslant 0$. Then k_1 and k_2 exist.

Proof. Let

$$k_1 > \max\left\{r_1, \frac{-v_1}{C_{11}}\right\}, \qquad k_2 > \max\left\{r_2, \left|\frac{C_{21}}{C_{22}}k_1 + \frac{v_2}{C_{22}}\right|\right\}$$

If $\rho_1 = k_1$, then $\rho_1 > -v_1/C_{11}$, so $-C_{11}\rho_1 < v_1$ and therefore

$$-v_1 - C_{11}\rho_1 - C_{12}\rho_2 < -C_{12}\rho_2 \leqslant 0.$$

Thus, $\dot{\rho}_1 < 0$. If $\rho_2 = k_2$ and $0 \leqslant \rho_1 \leqslant k_1$, then

$$\rho_2 > \left| \frac{C_{21}}{C_{22}} k_1 + \frac{v_2}{C_{22}} \right|$$

and so $-C_{22}\rho_2 < -|C_{21}k_1 + v_2|$. But if $C_{21} < 0$, then $-C_{21}\rho_1 \leqslant -C_{21}k_1$, so

$$-v_2 - C_{21}\rho_1 - C_{22}\rho_2 < -(v_2 + C_{21}k_1 + |C_{21}k_1 + v_2|) \leqslant 0$$

while if $C_{21} \geqslant 0$, then $-C_{21}\rho_1 \leqslant 0$, so

$$-v_2 - C_{21}\rho_1 - C_{22}\rho_2 < -(v_2 + |C_{21}k_1 + v_2|) = -(v_2 + C_{21}k_1 + v_2) \leqslant 0$$

So, if $\rho_2 = k_2$ and $0 \leqslant \rho_1 \leqslant k_1$, then $-v_2 - C_{21}\rho_1 - C_{22}\rho_2 < 0$, and then $\dot{\rho}_2 < 0$. ▮

Corollary. Let $C_{11} > 0$, $C_{22} > 0$, $C_{21} \geqslant 0$, $v_1 \geqslant 0$, $v_2 \leqslant 0$. Then k_1 and k_2 exist.

Lemma 5. Let $C_{11} > 0$, $C_{22} > 0$, $C_{12} < 0$, $C_{21} \geqslant 0$. Then k_1 and k_2 exist.

Proof. Let

$$k_1 > \max\left\{ r_1, \left| \frac{C_{12}}{C_{11}} k_2 + \frac{v_1}{C_{11}} \right| \right\}, \qquad k_2 > \max\left\{ r_2, \left| \frac{v_2}{C_{22}} \right| \right\}$$

If $\rho_1 = k_1$ and $0 \leqslant \rho_2 \leqslant k_2$, then

$$\rho_1 = k_1 > \left| \frac{C_{12}}{C_{11}} k_2 + \frac{v_1}{C_{11}} \right|$$

so $-C_{11}\rho_1 < |-C_{12}k_2 + v_1|$ and $-C_{12}\rho_2 \leqslant -C_{12}k_2$, hence

$$-v_1 - C_{11}\rho_1 - C_{12}\rho_2 < -(v_1 + C_{12}k_2 + |C_{12}k_2 + v_1|) \leqslant 0$$

and thus $\dot{\rho}_1 < 0$. Similarly, if $\rho_2 = k_2$ and $0 \leqslant \rho_1 \leqslant k_1$, then $\rho_2 = k_2 > |v_2/C_{22}|$, so $-C_{22}\rho_2 < -|v_2|$ and hence

$$-v_2 - C_{21}\rho_1 - C_{22}\rho_2 < -(v_2 + |v_2|) - C_{21}\rho_1 \leqslant 0$$

so $\dot{\rho}_2 < 0$. ▮

Corollary. Let $C_{11} > 0$, $C_{22} > 0$, $C_{12} \geqslant 0$, and $C_{21} < 0$. Then k_1 and k_2 exist.

Lemmas 1–5 and the associated corollaries include all possible combinations of physical values for C_{12}, C_{21}, v_1, and v_2 [i.e., those that satisfy conditions (i) and (ii) in Section 1]; hence, Proposition 7 is proved and there exists a compact, positive invariant set K of the form $\{(\rho_1, \rho_2) \mid 0 \leqslant \rho_1 \leqslant k_1, 0 \leqslant \rho_2 < k_2\}$, k_1, k_2 finite, for the flow in Q with $p \in K$.

3.2. Fixed Points on the Boundary

These results on compact, positive invariant sets and the Poincaré–Bendixson theorem will now be used to establish some results on the asymptotic behavior of the two-species gas mixture. The first results concern the cases where fixed points occur only on the boundary of Q. These cases are greatly simplified by the Poincaré index theorem,[8] which states that if a two-dimensional system has a closed orbit (corresponding to a periodic solution), then there must be a fixed point in the region bounded by this orbit. Since Q is invariant, by Proposition 1 this means that there can be no closed orbit in Q unless there is a fixed point in the interior of Q.

Before pursuing this, however, a useful result is stated:

Proposition 8. Recall that $C_{11} \geqslant 0$ and let $p = (r_1, 0)$ with $r_1 > 0$.

1. If $v_1 \geqslant 0$, but v_1 and C_{11} not both zero, then $\omega(p) = (0, 0)$.
2. If $v_1 < 0$, then $\omega(p) = (-v_1/C_{11}, 0)$.

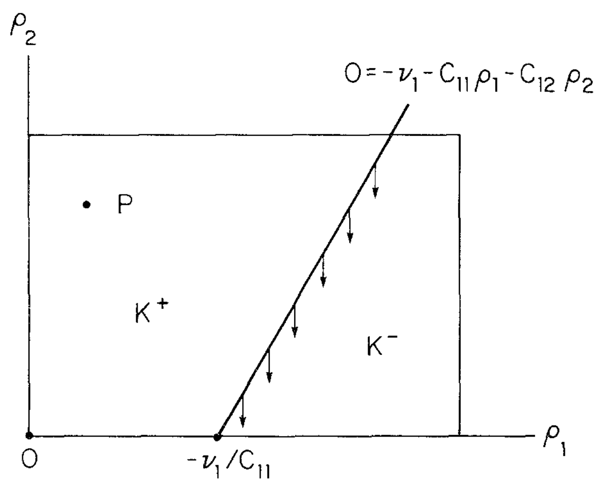

Fig. 6. The sets K^+ and K^- for $C_{12} < 0$, in which case K^- is positive invariant. For $C_{12} > 0$, K^+ is positive invariant, while for $C_{12} = 0$ both are positive invariant.

Proof. If $C_{11}=0$, then $\rho(t;p)=(r_1\exp(-v_1 t), 0)$. If $C_{11}>0$, there exists compact, positive invariant K by Proposition 7, so $\rho(t;p)$ exists for all $t\geqslant 0$ and $\omega(p)\neq\varnothing$. Since $\dot\rho_1=\rho_1(-v_1-C_{11}\rho_1)$, it follows that $\dot\rho_1<0$ for $v_1\geqslant 0$ and $\rho_1\neq 0$, while for $v_1<0$, one has $\dot\rho_1>0$ if $0<\rho_1<-v_1/C_{11}$ and $\dot\rho_1<0$ if $-v_1/C_{11}<\rho_1$. The proposition easily follows. ∎

Corollary. Recall that $C_{22}\geqslant 0$ and let $p=(0, r_2)$ with $r_2>0$.

1. If $v_2\geqslant 0$, but v_2 and C_{22} not both zero, then $\omega(p)=(0,0)$.
2. If $v_2<0$, then $\omega(p)=(0, -v_2/C_{22})$.

Because of this proposition, only the ω-limits of points $p=(r_1, r_2)$ with $r_1>0$ and $r_2>0$, points in the interior of Q, will need to be considered below.

Proposition 9. If $C_{11}>0$, $C_{22}>0$, $C_{21}\geqslant 0$, $v_1\geqslant 0$, and $v_2\geqslant 0$, then $(0,0)$ is the ω-limit of every point $p\in Q$.

Proof. Note that $-v_1/C_{11}\leqslant 0$ and $-v_2/C_{22}\leqslant 0$. Then by Proposition 2 the only fixed points other than $(0,0)$ that might be in Q satisfy

$$0=-v_1-C_{11}\bar\rho_1-C_{12}\bar\rho_2 \tag{38a}$$

$$0=-v_2-C_{21}\bar\rho_1-C_{22}\bar\rho_2 \tag{38b}$$

with $\bar\rho_1\neq 0$ and $\bar\rho_2\neq 0$. But if $\bar\rho_1>0$ and $\bar\rho_2>0$, then $-v_2-C_{21}\bar\rho_1-C_{22}\bar\rho_2<0$; hence, $(0,0)$ is the only fixed point in Q. By Proposition 7 there exists a compact, positive invariant set $K\subset Q$ with $(0,0)\in K$ and $p\in K$. Since $(0,0)$ is the only fixed point in K, and it is on the boundary, there can be no closed orbit in K (Poincaré index theorem). Thus, by the Poincaré–Bendixson theorem, $\omega(p)=(0,0)$. ∎

Corollary. If $C_{11}>0$, $C_{22}>0$, $C_{12}\geqslant 0$, $v_1\geqslant 0$, and $v_2\geqslant 0$, then $(0,0)$ is the ω-limit of every point $p\in Q$.

Proposition 10. If $C_{11}>0$, $C_{22}>0$, $v_1<0$, $v_2\geqslant 0$, and $v_1C_{21}-v_2C_{11}<0$, then $(-v_1/C_{11}, 0)$ is the ω-limit of every point p in the interior of Q.

Proof. Note that $(0,0)$ and $(v_1/C_{11}, 0)$ are both fixed points in Q, but that $-v_2/C_{22}\leqslant 0$. Solutions of Eqs. (38) with $\bar\rho_1\neq 0$ and $\bar\rho_2\neq 0$ might provide additional fixed points in Q. But for $\bar\rho_1>0$, $\bar\rho_2>0$, and $C_{21}\geqslant 0$, one has $-v_2-C_{21}\bar\rho_1-C_{22}\bar\rho_2<0$, so there is no such solution if $C_{21}\geqslant 0$. If $C_{21}<0$, then a solution requires

$$\bar\rho_2(C_{11}C_{22}-C_{12}C_{21})=(v_1C_{21}-v_2C_{11})<0$$

which can hold only if $\bar{\rho}_2 < 0$. Thus, $(0, 0)$ and $(-v_1/C_{11}, 0)$ are the only fixed points in Q. By Proposition 7 there exists a compact, positive invariant box $K \subset Q$ containing $(0, 0)$, $(-v_1/C_{11}, 0)$, and p. By the Poincaré index theorem, K does not contain any closed orbit.

Now consider the two sets

$$K^- = K \cap \{(\rho_1, \rho_2) \mid -v_1 - C_{11}\rho_1 - C_{12}\rho_2 \leqslant 0\} \tag{39}$$

$$K^+ = K \cap \{(\rho_1, \rho_2) \mid -v_1 - C_{11}\rho_1 - C_{12}\rho_2 \geqslant 0\} \tag{40}$$

See Figure (6). Both K^- and K^+ are compact and either K^- or K^+ is positive invariant, since on their mutual boundary, i.e., on $-v_1 - C_{11}\rho_1 - C_{12}\rho_2 = 0$, $\dot{\rho}_1 = 0$ and $\dot{\rho}_2$ does not change sign [otherwise there would be a fixed point other than $(0, 0)$ and $(-v_1/C_{11}, 0)$ in Q]. Suppose now that there exists an unbounded increasing sequence $\{t_n\}$ such that $\lim \rho(t_n; p) = (0, 0)$. Note that since $-v_1 > 0$ there is a neighborhood of $(0, 0)$ (in the Q relative topology) contained in K^+. Then there must exist N and M with $M > N$ such that $\rho_1(t_N)$, $\rho_1(t_M) \in K^+$ and $\rho_1(t_M) < \rho_1(t_N)$. By the mean value theorem $\rho_1(t_M) - \rho_1(t_N) = \dot{\rho}_1(\tau)(t_M - t_N)$, where $t_N \leqslant \tau \leqslant t_M$; thus $\dot{\rho}_1(\tau) < 0$, so $\rho(\tau; p) \in K^-$. This contradicts the positive invariance either of K^+ or of K^-. Thus $(0, 0) \notin \omega(p)$, and so by the Poincaré–Bendixson theorem $\omega(p) = (-v_1/C_{11}, 0)$. ∎

This proposition can be extended slightly to the case $C_{11} > 0$, $C_{22} > 0$, $v_1 < 0$, $v_2 \geqslant 0$, $C_{11}C_{22} - C_{12}C_{21} \neq 0$, and $v_1 C_{21} - v_2 C_{11} = 0$. Then $(-v_1/C_{11}, 0)$ is still the ω-limit of all points in the interior of Q even though one of its stability-determining eigenvalues is zero and $(-v_1/C_{11}, 0)$ is a double fixed point [i.e., a twofold degenerate root of Eqs. (38)]. The proof is essentially identical.

Again, permuting indicies, there is an obvious corollary:

Corollary. If $C_{11} > 0$, $C_{22} > 0$, $v_1 \geqslant 0$, $v_2 < 0$, and $v_2 C_{12} - v_1 C_{22} < 0$, then $(0, -v_2/C_{22})$ is the ω-limit of every point p in the interior of Q; and similarly for $C_{11} > 0$, $C_{22} > 0$, $v_1 \geqslant 0$, $v_2 < 0$, $C_{11}C_{22} - C_{12}C_{21} \neq 0$, and $v_2 C_{12} - v_1 C_{22} = 0$.

There is one more case in which fixed points appear only on the boundary of Q.

Proposition 11. If $C_{11} > 0$, $C_{22} > 0$, $v_1 < 0$, $v_2 < 0$, $v_1 C_{21} - v_2 C_{11} < 0$, and $v_2 C_{12} - v_1 C_{22} > 0$, then $(-v_1/C_{11}, 0)$ is the ω-limit of every point p in the interior of Q:

Proof. Note that $(0, 0)$, $(-v_1/C_{11}, 0)$, and $(0, -v_2/C_{22})$ are all fixed points in Q. Any other fixed points must satisfy Eqs. (38) with $\bar{\rho}_1 > 0$ and $\bar{\rho}_2 > 0$. Thus,

$$(v_1 C_{21} - v_2 C_{11}) + \bar{\rho}_2(C_{12} C_{21} - C_{11} C_{22}) = 0 \tag{41a}$$

$$(v_2 C_{12} - v_1 C_{22}) + \bar{\rho}_1(C_{12} C_{21} - C_{11} C_{22}) = 0 \tag{41b}$$

which has no solutions in Q. The remainder of the proof is exactly as before. If either $(0, 0)$ or $(0, -v_2/C_{22})$ were in the ω-limit set of p, then the positive invariance of K^+ or K^- would be contradicted. ▮

This proposition also can be supplemented. If $C_{11} > 0$, $C_{22} > 0$, $v_1 < 0$, $v_2 < 0$, $v_1 C_{21} - v_2 C_{11} = 0$, $v_2 C_{12} - v_1 C_{22} > 0$ (or $v_1 C_{21} - v_2 C_{11} < 0$ and $v_2 C_{12} - v_1 C_{22} = 0$) and $C_{11} C_{22} - C_{21} C_{12} \neq 0$, then $(-v_1/C_{11}, 0)$ is the ω-limit of all points in the interior of Q. Again there are corollaries obtained by permuting indices.

Corollary. If $C_{11} > 0$, $C_{22} > 0$, $v_1 < 0$, $v_2 < 0$, $v_1 C_{21} - v_2 C_{11} > 0$, and $v_2 C_{12} - v_1 C_{22} < 0$, then $(0, -v_2/C_{22})$ is the ω-limit of every point p in the interior of Q. Also, if $v_2 C_{12} - v_1 C_{22} = 0$, $v_1 C_{21} - v_2 C_{11} > 0$ (or $v_2 C_{12} - v_1 C_{22} < 0$, $v_1 C_{21} - v_2 C_{11} = 0$) and $C_{11} C_{22} - C_{12} C_{21} \neq 0$, then $(0, -v_2/C_{22})$ is the ω-limit of every point p in the interior of Q.

Propositions 8–11 and the associated corollaries exhaust all of the possible ω-limit sets when $C_{11} > 0$, $C_{22} > 0$, and there are no fixed points in the interior of Q. In each case one of the fixed points on the boundary attracts every initial condition in the interior of Q. In these parameter ranges a gas mixture described by Eqs. (1) with $N = 2$ evolves in such a way that either both species are consumed (absorbed) completely by the background gas (Propositions 9 and corollary) or one species is consumed completely by the other and the background gas (Propositions 10 and 11 and corollaries). This concludes the study of two-species systems with fixed points on the boundary of Q. Depending on the parameter values, fixed points in the interior of Q also can exist. This case is treated next.

3.3. Fixed Points in the Interior of Q

The somewhat more complicated and dynamically interesting cases that occur when there is a fixed point in the interior of Q will now be examined. A fixed point in the interior of Q can complicate the dynamics either by providing an attractor on which both number densities are positive or by breaking Q into disjoint basins of attraction for fixed points on the boundary of Q. This richer dynamical behavior is reflected in the facts that the proofs require more constructions and the parameter ranges included are not quite complete.

Consider the parameter range described by $C_{11} > 0$, $C_{22} > 0$, $C_{12} > 0$, $C_{21} < 0$, $C_{12}v_2 - v_1 C_{22} > 0$, and $C_{21}v_1 - v_2 C_{11} > 0$. These conditions imply that $v_1 < 0$ and that there are at least three fixed points in Q, specifically $(0, 0)$, $(-v_1/C_{11}, 0)$, and $(\bar{\rho}_1, \bar{\rho}_2)$, with

$$\bar{\rho}_1 = \frac{C_{12}v_2 - v_1 C_{22}}{C_{11}C_{22} - C_{12}C_{21}} \tag{42a}$$

$$\bar{\rho}_2 = \frac{C_{21}v_1 - v_2 C_{11}}{C_{11}C_{22} - C_{12}C_{21}} \tag{42b}$$

(see Table II). Also, only the fixed point at $(\bar{\rho}_1, \bar{\rho}_2)$ is stable. The fixed point $(0, -v_2/C_{22})$ is in Q or not in Q depending upon v_2 being negative or positive.

Now define the lines (see Fig. 7)

$$l_1 = \{(\rho_1, \rho_2) \mid 0 = -v_1 - C_{11}\rho_1 - C_{12}\rho_2\} \tag{43}$$

$$l_2 = \{(\rho_1, \rho_2) \mid 0 = -v_2 - C_{21}\rho_1 - C_{22}\rho_2\} \tag{44}$$

and the line segment $S \subset l_1$

$$S = \{(\rho_1, \rho_2) \in l_1 \mid \bar{\rho}_1 \leqslant \rho_1 \leqslant (-v_1/C_{11}), 0 \leqslant \rho_2 \leqslant \bar{\rho}_2\} \tag{45}$$

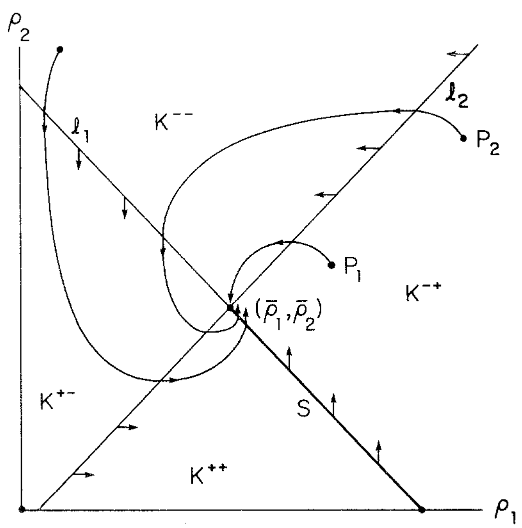

Fig. 7. The geometrical content of Lemma 6 for the case $v_2 > 0$. The trajectory of P_1 enters K^{--}, but never exits, $\omega(P_1) = (\bar{\rho}_1, \bar{\rho}_2)$. The trajectory of P_2 passes through K^{--}, K^{+-}, K^{++} and crosses S.

Note that $(-v_1/C_{11}, 0) \in l_1$, $(\bar{\rho}_1, \bar{\rho}_2) \in l_1$, $(\bar{\rho}_1, \bar{\rho}_2) \in l_2$, and that $(\bar{\rho}_1, \bar{\rho}_2) = l_1 \cap l_2$, since $C_{11}C_{22} - C_{12}C_{21} > 0$.

Further, define $L_1(\rho_1, \rho_2) = -v_1 - C_{11}\rho_1 - C_{12}\rho_2$, $L_2(\rho_1, \rho_2) = -v_2 - C_{21}\rho_1 - C_{22}\rho_2$, and the sets

$$K^{++} = \{(\rho_1, \rho_2) \mid L_1(\rho_1, \rho_2) \geqslant 0, L_2(\rho_1, \rho_2) \geqslant 0\} \cap K \tag{46}$$

$$K^{+-} = \{(\rho_1, \rho_2) \mid L_1(\rho_1, \rho_2) \geqslant 0, L_2(\rho_1, \rho_2) \leqslant 0\} \cap K \tag{47}$$

$$K^{-+} = \{(\rho_1, \rho_2) \mid L_1(\rho_1, \rho_2) \leqslant 0, L_2(\rho_1, \rho_2) \geqslant 0\} \cap K \tag{48}$$

$$K^{--} = \{(\rho_1, \rho_2) \mid L_1(\rho_1, \rho_2) \leqslant 0, L_2(\rho_1, \rho_2) \leqslant 0\} \cap K \tag{49}$$

where K is a compact, positive invariant set containing any initial condition p and all the fixed points in Q. See Fig. 7. Note that $K = K^{++} \cup K^{+-} \cup K^{-+} \cup K^{--}$ and that none of these sets is empty, because $\bar{\rho}_1 > 0$ and $\bar{\rho}_2 > 0$. Each of these sets is compact and can have fixed points only on its boundaries and thus cannot contain a closed orbit.

Lemma 6. If $C_{11} > 0$, $C_{22} > 0$, $C_{12} > 0$, $C_{21} < 0$, $C_{12}v_2 - v_1C_{22} > 0$, and $C_{21}v_1 - v_2C_{11} > 0$, then the trajectory of a point p in the interior of Q either crosses S or has $\omega(p) = (\bar{\rho}_1, \bar{\rho}_2)$.

Proof. The sets K^{++}, K^{-+}, K^{+-}, and K^{--} are not positive invariant. In fact, consider the mutual boundary of K^{++} and K^{-+} given by

$$L_1(\rho_1, \rho_2) = 0, \qquad L_2(\rho_1, \rho_2) \geqslant 0 \tag{50}$$

This is just the segment S. On this segment $\dot{\rho}_1 = 0$ and $\dot{\rho}_2 \geqslant 0$. Now, in K^{++}, $L_1(\rho_1, \rho_2) \geqslant 0$ implies, since $C_{12} > 0$, that

$$\rho_2 \leqslant \frac{-v_1}{C_{12}} - \frac{C_{11}}{C_{12}}\rho_1 \tag{51}$$

Thus, $\dot{\rho}_2 \geqslant 0$ on the boundary with K^{-+} implies that the flow is out of K^{++} into K^{-+} on this segment S. Similar considerations at each of the four mutual boundaries establishes that the flow is such that if a trajectory exits K^{++}, it enters K^{-+}. If a trajectory exits K^{-+}, it enters K^{--}. If a trajectory exits K^{--}, it enters K^{+-}. Finally, if a trajectory exits K^{+-}, it enters K^{++}.

Suppose that the trajectory of p is entirely contained in one of the sets K^{++}, K^{-+}, K^{--}, or K^{+-} and that $\omega(p) \neq (\bar{\rho}_1, \bar{\rho}_2)$. Since $(\bar{\rho}_1, \bar{\rho}_2)$ is asymptotically stable, it follows that $(\bar{\rho}_1, \bar{\rho}_2) \notin \omega(p)$. But since the trajectory of p remains in a compact set for all time, the Poincaré–Bendixson theorem applies and since there are no periodic orbits in any of the sets it must be that $\omega(p)$ contains one of the other fixed points. If the set contains no fixed point other than $(\bar{\rho}_1, \bar{\rho}_2)$, a contradiction arises. If the set contains one or more of $(0, 0)$, $(-v_1/C_{11}, 0)$, or $(0, -v_2/C_{22})$, then one of these is in $\omega(p)$. Suppose $(0, -v_2/C_{22}) \in \omega(p)$. Then there exists $\{t_n\}$ unbounded and increasing such that $\rho_1(t_n) \to 0$. But

$$L_1(0, -v_2/C_{22}) = (-v_1 C_{22} + v_2 C_{12})/C_{22} > 0 \tag{52}$$

so there exists U, a Q-neighborhood of $(0, -v_2/C_{22})$, in which $\dot{\rho}_1 \geqslant 0$, i.e., in which $L_1(\rho_1, \rho_2) > 0$. Since $\lim \rho_1(t_n) = 0$, there exist N and M with $N < M$ such that $\rho(t_N; p) \in U$ and $\rho(t_M; p) \in U$ and $\rho_1(t_M) < \rho_1(t_N)$. Then the mean value theorem gives

$$\rho_1(t_M) - \rho_1(t_N) = \dot{\rho}_1(\tau)(t_M - t_N) \tag{53}$$

with $t_N \leqslant \tau \leqslant t_M$ and thus $\dot{\rho}_1(\tau) < 0$. But this requires the trajectory to exit whatever K it is in. A similar argument works at the other two fixed points and a contradiction arises.

Thus, the assertion that the trajectory of p lies entirely in one of K^{++}, K^{-+}, K^{--}, or K^{+-} and $(\bar{\rho}_1, \bar{\rho}_2) \notin \omega(p)$ contradicts the Poincaré–Bendixson theorem. Thus, either the trajectory of p is entirely in one of these sets and $\omega(p) = (\bar{\rho}_1, \bar{\rho}_2)$ or the trajectory exits the set. Bur since p is arbitrary, it follows that if a trajectory intersects one of K^{++}, K^{-+}, K^{--}, or K^{+-}, then either its ω-limit is $(\bar{\rho}_1, \bar{\rho}_2)$ or it exits the set and enters the next in the list.

Since S is the boundary between K^{++} and K^{-+}, the lemma follows. ∎

The content of this geometrically obvious lemma is shown in Fig. 7. Note that even if the trajectory of p intersects S an infinite number of times, it could still be that $\omega(p) = (\bar{\rho}_1, \bar{\rho}_2)$. In fact, this is almost certainly the case, as will be shown below for certain parameter regimes.

Proposition 12. If $C_{11} > 0$, $C_{22} > 0$, $C_{12} > 0$, $C_{21} < 0$, $C_{12} v_2 - v_1 C_{22} > 0$, and $C_{21} v_1 - v_2 C_{11} > 0$, then the ω-limit of every point p in the interior of Q is either $(\bar{\rho}_1, \bar{\rho}_2)$ or a closed orbit.

Proof. By Proposition 7 there is a compact, positive invariant set K containing p and all the fixed points. To establish the proposition, it is then only necessary to show that $(0, 0)$, $(-v_1/C_{11}, 0)$, and, if $v_2 < 0$, $(0, -v_2/C_{22})$ are not in the ω-limit of p. At the fixed point $(-v_1/C_{11}, 0)$ the Jacobian is

$$J(-v_1/C_{11}, 0) = \begin{bmatrix} v_1 & C_{12}(v_1/C_{11}) \\ 0 & (C_{21} v_1 - v_2 C_{11})/C_{11} \end{bmatrix} \tag{54}$$

which has positive eigenvalue $(C_{21}v_1 - v_2 C_{11})/C_{11}$ with an eigenvector of the form $(\gamma, 1)$. By the stable manifold theorem[8] there is a local one-dimensional unstable manifold tangent to $(\gamma, 1)$. Let p_0 be on this manifold. By Proposition 7 it is possible to choose K containing p_0 as well; thus, p_0 has a positive trajectory in K for all time, $t \geqslant 0$, and either $\omega(p_0) = (\bar{\rho}_1, \bar{\rho}_2)$ or else the trajectory of p_0 intersects S (Lemma 6). Since p_0 is on the unstable manifold of $(-v_1/C_{11}, 0)$, it has a trajectory for all $t \leqslant 0$ as well, and $\rho(t; p_0) \to (-v_1/C_{11}, 0)$ as $t \to -\infty$. If the forward trajectory of p_0 intersects S, denote its first intersection point by $(\hat{\rho}_1, \hat{\rho}_2)$. If it does not intersect S, then write $(\hat{\rho}_1, \hat{\rho}_2) = (\bar{\rho}_1, \bar{\rho}_2)$.

Now, let C be the closed curve defined by the (forward and backward) trajectory of p_0 and the line segment on S from $(-v_1/C_{11}, 0)$ to $(\hat{\rho}_1, \hat{\rho}_2)$ (Fig. 8). This curve C separates K into two regions (Jordan curve Lemma) and the interior region enclosed by C is positive invariant because no trajectory can cross the unstable manifold (trajectory of p_0) and the flow across S is into this region. This and Lemma 6 ensure that $(\bar{\rho}_1, \bar{\rho}_2)$ is inside C and that every point on S is either inside C or on its boundary. For any point p in the interior of Q, Lemma 6 then guarantees that $\omega(p)$ is contained in the compact, positive invariant set bounded by C.

The positive ρ_1 axis is the unique stable manifold for $(-v_1/C_{11}, 0)$ and so, by the stable manifold theorem, $\omega(p) \neq \{(-v_1/C_{11}, 0)\}$ for any p not

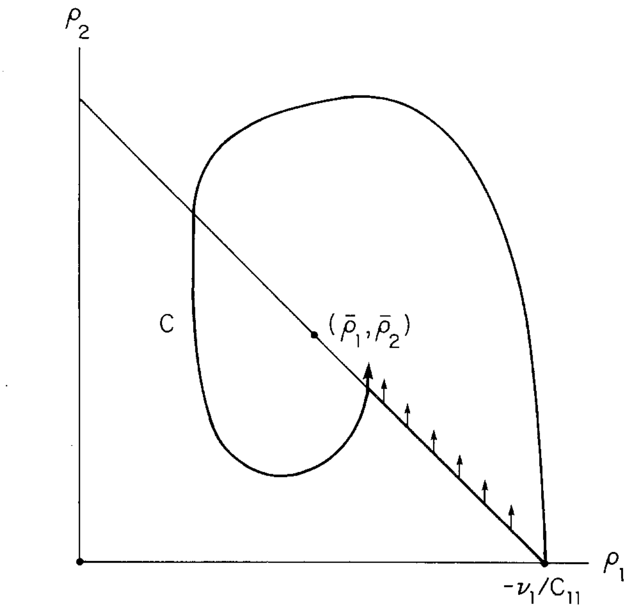

Fig. 8. The closed curve C generated by S and the global unstable manifold of $(-v_1/C_{11}, 0)$.

on this axis (e.g., for points p in the interior of Q). Further, if $(-\nu_1/C_{11}, 0) \in \omega(p)$ the Poincaré–Bendixson theorem requires that $(\bar{\rho}_1, \bar{\rho}_2) \in \omega(p)$ also. But this would imply that $\omega(p) = (\bar{\rho}_1, \bar{\rho}_2)$, since this latter fixed point is asymptotically stable. It follows that $(-\nu_1/C_{11}, 0)$ is not in the ω-limit of any point p in the interior of Q. The Poincaré–Bendixson then requires that $\omega(p)$ is either a closed orbit contained inside C or else is $(\bar{\rho}_1, \bar{\rho}_2)$ itself. ▮

It should be obvious from this that the ω-limit of the global unstable manifold (the trajectory of any p_0 on the local unstable manifold) in fact determines the ω-limit of all points exterior to C. In fact, if the ω-limit of p_0 is a closed orbit, then all points outside that closed orbit tend to it. And if $\omega(p_0) = (\bar{\rho}_1, \bar{\rho}_2)$, then $\omega(p) = (\bar{\rho}_1, \bar{\rho}_2)$ for all p in the interior of Q.

The occurrence of closed orbits can be ruled out for certain parameter ranges by constructing a bounding map on a map generated by the flow of Eqs. (36). To do this, let the segment S be parametrized as

$$s \mapsto (\bar{\rho}_1 + s, \bar{\rho}_2 - (C_{11}/C_{12})\, s) \tag{55}$$

with $0 \leqslant s \leqslant (C_{12}/C_{11})\, \bar{\rho}_2$. A linear map $T: S \to S$ will now be constructed (see Fig. 9). From the point s_0, corresponding to (ρ_1^0, ρ_2^0) on the segment

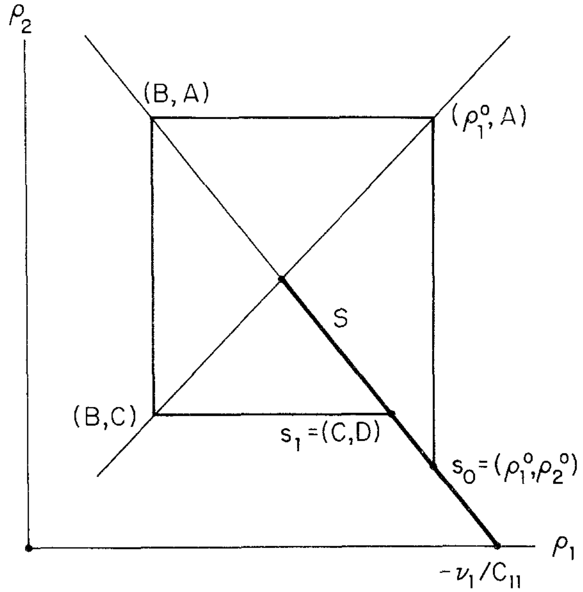

Fig. 9. Construction of the map $T: S \to S$, $T(s_0) = s_1$.

$S \subset l_1$ draw a vertical line segment to the point (ρ_1^0, A) on the line l_2. From (ρ_1^0, A) draw a horizontal segment to the point (B, A) on l_1 and then a vertical segment to (B, C) on l_2. Finally, draw a horizontal segment from (B, C) to (D, C) on S. The point (D, C) corresponds to a parameter value s_1, $T(s_0) = s_1$ (Fig. 9). Some straightforward algebra gives the map T as

$$T(s) = \left(\frac{C_{12}C_{21}}{C_{11}C_{22}}\right)^2 s \tag{56}$$

Now, by Lemma 6, the flow of Eqs. (36) generates a map $P\colon S \to S$ with $s = 0$ and $s = (C_{12}/C_{11})\,\bar{\rho}_2$ as fixed points. There is a closed orbit (periodic solution) for the system if and only if P has another fixed point in the open segment $0 < s < (C_{12}/C_{11})\,\bar{\rho}_2$. The map P may be compared to T by noting that

$$\rho_1^0(-\nu_1 - C_{11}\rho_1^0 - C_{12}\rho_2) < 0, \qquad \rho_2^0 < \rho_2 \leqslant A \tag{57}$$

$$A(-\nu_2 - C_{21}\rho_1 - C_{22}A) < 0, \qquad B \leqslant \rho_1 < \rho_1^0 \tag{58}$$

$$B(-\nu_1 - C_{11}B - C_{12}\rho_2) > 0, \qquad C \leqslant \rho_2 < A \tag{59}$$

$$C(-\nu_2 - C_{21}\rho_1 - C_{22}C) > 0, \qquad B < \rho_1 \leqslant D \tag{60}$$

These inequalities ensure that, for $0 < s < (C_{12}/C_{11})\,\bar{\rho}_2$,

$$T(s) > P(s) \tag{61}$$

(see Fig. 10). Thus, P has a fixed point $s_F > 0$ only if

$$T(s_F) > s_F \tag{62}$$

Therefore, if

$$T(s) \leqslant s \tag{63}$$

for all s, P cannot have a fixed point other than $s = 0$ and $s = (C_{12}/C_{11})\,\bar{\rho}_2$. Thus, if,

$$(C_{12}C_{21}/C_{11}C_{22})^2 \leqslant 1 \tag{64}$$

there can be no closed orbit. In this case $(\bar{\rho}_1, \bar{\rho}_2)$ is the ω-limit of all points in the interior of Q.

Again there is an obvious corollary to Proposition 12 obtained by permuting indices and the condition (64) holds for this case also.

Two parameter ranges remain, both characterized by $C_{11} > 0$, $C_{22} > 0$, $C_{12} > 0$, and $C_{21} > 0$. They are distinguished by $C_{11}C_{22} - C_{12}C_{21} > 0$,

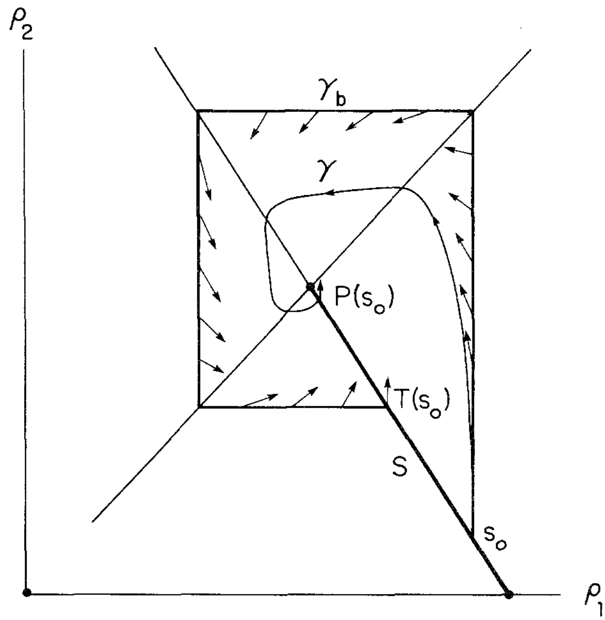

Fig. 10. Comparison of $T(s_0)$ and the exact Poincaré map $P(s_0)$. The true trajectory γ is bounded by the approximate trajectory γ_b.

$C_{12}v_2 - v_1 C_{22} > 0$, and $C_{21}v_1 - v_2 C_{11} > 0$ versus $C_{11}C_{22} - C_{12}C_{21} < 0$, $C_{12}v_2 - v_1 C_{22} < 0$, and $C_{21}v_1 - v_2 C_{11} < 0$. A bit of manipulation reveals that either case implies that $v_1 < 0$ and $v_2 < 0$, so all four fixed points are in Q. The flows for these parameter ranges will be seen to be quite different from the flow discussed in the paragraph above and depicted in Figs. 7–10. To study these final parameter ranges, a lemma similar to one used by Hirsch and Smale[2] in a similar context will be needed.

Lemma 7. Let $C_{11} > 0$, $C_{22} > 0$, $C_{12} > 0$, $C_{21} > 0$, and either:

(a) $C_{11}C_{22} - C_{12}C_{21} > 0$, $C_{12}v_2 - v_1 C_{22} > 0$, and $C_{21}v_1 - v_2 C_{11} > 0$

(b) $C_{11}C_{22} - C_{12}C_{21} < 0$, $C_{12}v_2 - v_1 C_{22} < 0$, and $C_{21}v_1 - v_2 C_{11} < 0$.

Let K be any compact, positive invariant set containing all four fixed points and define K^{++}, K^{+-}, K^{-+}, and K^{--} as above (see Fig. 11). Then K^{+-} and K^{-+} are positive invariant.

Proof. It will be proven that K^{+-} is positive invariant; the proof for K^{-+} is completely analogous. In K^{+-} the conditions

$$-v_1 - C_{11}\rho_1 - C_{12}\rho_2 \geqslant 0 \tag{65a}$$

$$-v_2 - C_{21}\rho_1 - C_{22}\rho_2 \leqslant 0 \tag{65b}$$

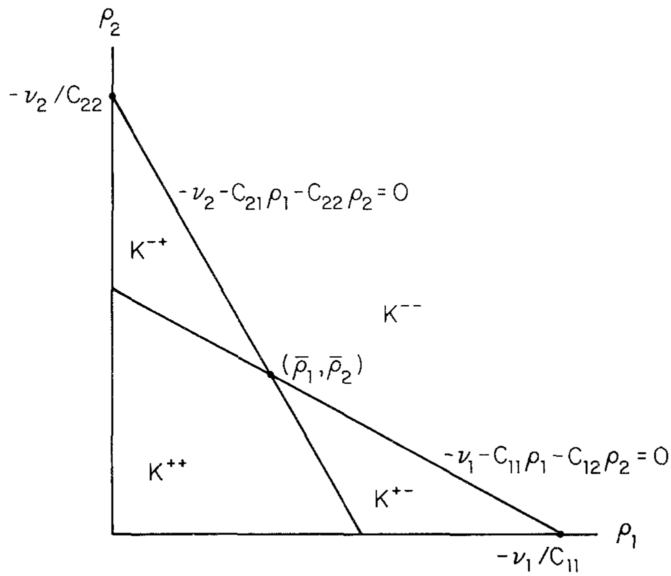

Fig. 11. The four sets K^{++}, K^{+-}, K^{-+}, and K^{--}. The figure is for case (a) of Lemma 7.

are satisfied. At $-v_1-C_{11}\rho_1-C_{12}\rho_2=0$, K^{+-} bounds on K^{--}. Since $C_{12}>0$,

$$\frac{-v_1}{C_{12}}-\frac{C_{11}}{C_{12}}\rho_1\geqslant\rho_2 \tag{66}$$

for $(\rho_1,\rho_2)\in K^{+-}$. But on the boundary with K^{--}, $\dot{\rho}_1=0$ and $\dot{\rho}_2\leqslant 0$, so there is no flow from K^{+-} into K^{--}. Similarly, at $-v_2-C_{21}\rho_1-C_{22}\rho_2=0$, K^{+-} bounds K^{++}, and since $C_{21}>0$

$$\frac{-v_2}{C_{21}}-\frac{C_{22}}{C_{21}}\rho_2\leqslant\rho_1 \tag{67}$$

for $(\rho_1,\rho_2)\in K^{+-}$. But on the boundary with K^{++}, $\dot{\rho}_1\geqslant 0$ and $\dot{\rho}_2=0$, so there is no flow from K^{+-} into K^{--}. Finally, K^{+-} can have no other boundaries other than part of the boundary of K . Since K is positive invariant, so is K^{+-}. ∎

This leads immediately to the following result.

Proposition 13. If $C_{11}>0$, $C_{22}>0$, $C_{12}>0$, $C_{21}>0$, $C_{11}C_{22}-C_{12}C_{21}>0$, $C_{12}v_2-v_1C_{22}>0$, and $C_{21}v_1-v_2C_{11}>0$, then $(\bar{\rho}_1,\bar{\rho}_2)$ is the ω-limit of every point p in the interior of Q.

Proof. By Proposition 7 there is a compact, positive invariant set K containing p and all four fixed points. There can be no periodic orbit in K, since any such orbit must encircle the fixed point $(\bar{\rho}_1, \bar{\rho}_2)$ and therefore intersect both K^{-+} and K^{+-}. But both are positive invariant by Lemma 7 and they intersect only at $(\bar{\rho}_1, \bar{\rho}_2)$; hence, there cannot be a periodic orbit.

Now suppose that $(-v_1/C_{11}, 0) \in \omega(p)$. It will be shown that this leads to a contradiction. Note that there is a neighborhood U of $(-v_1/C_{11}, 0)$ in which $\rho_2(-v_2 - C_{21}\rho_1 - C_{22}\rho_2) \geqslant 0$, since $C_{21}v_1 - v_2 C_{11} > 0$ and that equality holds only for $\rho_2 = 0$. From the supposition there must exist an unbounded, increasing sequence $\{t_n\}$ such that $\lim \rho_2(t_n, p) = 0$. Thus, for L large enough, $\rho(t_n; p) \in U$ for all $n > L$. This implies that $\dot{\rho}_2(t_n; p) > 0$ for all $n > L$ [with strict inequality, since $\rho_2(t_n; p) > 0$ by Proposition 1]. Further, there exist N and M with $M > N > L$ such that $\rho_2(t_M; p) < \rho_2(t_N; p)$. Then, by the mean value theorem,

$$\rho_2(t_M; p) - \rho_2(t_N; p) = \dot{\rho}_2(\tau; p)(t_M - t_N) \tag{68}$$

with $t_N < \tau < t_M$ and $\dot{\rho}_2(\tau; p) < 0$. Thus, either $\rho(\tau; p) \in K^{+-}$ or $\rho(\tau; p) \in K^{--}$, since it is only in these sets that $\dot{\rho}_2 \leqslant 0$. Also, since $\dot{\rho}_2(t; p)$ is continuous in t and $\dot{\rho}_2(t_N; p) > 0$, there must exist τ' with $t_N < \tau' < \tau$ such that $\dot{\rho}_2(\tau'; p) = 0$ [but note that $\rho(\tau'; p) \neq \bar{\rho}$], so that $\rho(\tau'; p) \in K^{+-}$ or $\rho(\tau'; p) \in K^{-+}$ (since $\dot{\rho}_2 = 0$ only on the boundaries of these sets). Note finally that

$$K^{-+} \cap K^{+-} = (\bar{\rho}_1, \bar{\rho}_2)$$

$$K^{-+} \cap K^{--} \subset \{(\rho_1, \rho_2) \in K \mid -v_2 - C_{21}\rho_1 - C_{22}\rho_2 = 0\}$$

$$K^{+-} \cap U = \varnothing$$

Thus, if $\rho(\tau'; p) \in K^{+-}$, and since $\rho(t_M; p) \in U$, the positive invariance of K^{+-} (Lemma 7) is contradicted. But if $\rho(\tau'; p) \in K^{-+}$ and $\rho(\tau; p) \in K^{+-}$, the positive invariance of K^{-+} is contradicted, while if $\rho(\tau'; p) \in K^{-+}$ and $\rho(\tau; p) \in K^{--}$ [recalling that $\dot{\rho}_2(\tau; p) < 0$, so $-v_2 - C_{21}\rho_1(\tau; p) - C_{22}\rho_2(\tau; p) \neq 0$], the positive invariance of K^{-+} is contradicted. Thus, the hypothesis $(-v_1/C_{11}, 0) \in \omega(p)$ contradicts Lemma 7.

By similar reasoning the fixed points $(0, 0)$ and $(0, -v_2/C_{22})$ are not in $\omega(p)$. Thus, by the Poincaré–Bendixson theorem, $\omega(p) = \{(\bar{\rho}_1, \bar{\rho}_2)\}$. ∎

Only one case remains, namely $C_{11} > 0$, $C_{22} > 0$, $C_{12} > 0$, $C_{21} > 0$, $C_{11}C_{22} - C_{12}C_{21} < 0$, $C_{12}v_2 - v_1 C_{22} < 0$, and $C_{21}v_1 - v_2 C_{11} < 0$. As mentioned previously, these conditions imply that $v_1 < 0$ and $v_2 < 0$. It is verified easily from Table II that $(0, 0)$ is unstable and that $(-v_1/C_{11}, 0)$ and $(0, -v_2/C_{22})$ are asympotically stable. From Proposition 4 and Eq. (15) it is clear that $(\bar{\rho}_1, \bar{\rho}_2)$ is a saddle point. There is thus an open set

U containing $(-v_1/C_{11}, 0)$ such that $\omega(p) = \{(-v_1/C_{11} 0)\}$ for every $p \in U$ and an open set V containing $(0, -v_2/C_{22})$ such that $\omega(p) = \{(0, -v_2/C_{22})\}$ for every $p \in V$. Because $(\bar{\rho}_1, \bar{\rho}_2)$ is a saddle point, it has local one-dimensional stable and unstable manifolds (stable manifold theorem), which generate global stable and unstable manifolds W^s and W^u by extending the local manifolds backward and forward in time, respectively. By definition $\omega(p) = \{(\bar{\rho}_1, \bar{\rho}_2)\}$ if and only if $p \in W^s$. Also, $\lim_{t \to -\infty} \rho(t; p) = (\bar{\rho}_1, \bar{\rho}_2)$ if and only if $p \in W^u$. Thus, there is an orbit connecting $(\bar{\rho}_1, \bar{\rho}_2)$ to itself if and only if $W^s \cap W^u \neq \emptyset$. Showing that this is not the case is the key to understanding the ω-limit sets of points in Q for this final parameter range.

Proposition 14. Let $C_{11} > 0$, $C_{22} > 0$, $C_{12} > 0$, $C_{21} > 0$, $C_{11} C_{22} - C_{12} C_{21} < 0$, $C_{12} v_2 - v_1 C_{22} < 0$, and $C_{21} v_1 - v_2 C_{11} < 0$. Also, let p be any point in the interior of Q but $p \notin W^s$. Then either $\omega(p) = \{(-v_1/C_{11}, 0)\}$ or $\omega(p) = \{(0, -v_2/C_{22})\}$. Also, let $q \in W^s$. Then $\omega(q) = \{(\bar{\rho}_1, \bar{\rho}_2)\}$, but in every neighborhood of q there exist p_1 and p_2 such that $\omega(p_1) = \{(-v_1/C_{11}, 0)\}$ and $\omega(p_2) = \{(0, -v_2 C_{22})\}$.

Proof. By Proposition 7 there exists a compact, positive invariant set K containing all four fixed points and p. By Lemma 7 and methods used in the preceding proofs there is no periodic orbit and $(0, 0) \notin \omega(p)$. Since $(-v_1/C_{11}, 0)$ and $(0, -v_2/C_{22})$ are asymptotically stable, if $(-v_1/C_{11}, 0) \in \omega(p)$, then in fact $\omega(p) = \{(-v_1/C_{11}, 0)\}$ and similarly for $(0, -v/C_{22})$. Since $p \notin W^s$, $\omega(p) \neq \{(\bar{\rho}_1, \bar{\rho}_2)\}$. Thus, by the Poincaré–Bendixson theorem, if $(\bar{\rho}_1, \bar{\rho}_2) \in \omega(p)$, there is an orbit connecting $(\bar{\rho}_1, \bar{\rho}_2)$ to itself. But any such orbit must intersect the boundary of $K^{+-} \cup K^{-+}$ twice, and both intersections would be transversal; the existence of such an orbit would therefore contradict Lemma 7. The first part of the proposition follows.

Now define the basin of attraction BA1 of the fixed point $(-v_1/C_{11}, 0)$ as the set of points in Q with omega-limit set $\{(-v_1/C_{11}, 0)\}$, and similarly define the basin of attraction BA2 of the fixed point $(0, -v_2/C_{22})$. The continuity of the flow ensures that BA1 and BA2 are open, and of course they are disjoint. Since Q is connected, it follows that any path from $(-v_1/C_{11}, 0)$ to $(0, -v_2/C_{22})$ must intersect $W^s \cup \{(0, 0)\}$. Thus, any path from $(-v_1/C_{11}, 0)$ to $(0, -v_2/C_{22})$ that intersects the coordinate axes only at these fixed points must intersect W^s. Since any such path may be continuously deformed to the coordinate axes, W^s must have an accumulation point p' on the axes. Note that $W^s \cup \{(0, 0)\}$ is closed and that, by Proposition 1, W^s does not intersect the axes. Thus, $p' = (0, 0)$.

The one-dimensional stable manifold is the union of $(\bar{\rho}_1, \bar{\rho}_2)$ and two trajectories. It was just shown that one of these trajectories connects $(0, 0)$

to $(\bar{\rho}_1, \bar{\rho}_2)$. By applying the Poincaré–Bendixson theorem to the time-reversed flow, it is easy to see that the other trajectory could either exit any compact set (in finite time) or else connect $(0, 0)$ to $(\bar{\rho}_1, \bar{\rho}_2)$. But this latter case would imply that W^s bounds an invariant region that contains only the fixed points $(0, 0)$ and $(\bar{\rho}_1, \bar{\rho}_2)$ and no closed orbits. Since there is no orbit connecting $(\bar{\rho}_1, \bar{\rho}_2)$ to itself, the omega-limits of all points inside this compact invariant region would be empty; this is ridiculous. Therefore, the second trajectory in W^s must exit every compact set.

It is then clear that W^s is a separatrix (it splits Q into two invariant regions) between BA1 and BA2. Since W^s has empty interior, the second part of the proposition follows. ∎

3.4. Bifurcation Sets and Phase Portraits

The asymptotic behavior of every trajectory in Q has been established in the previous subsections. Hence, it now is possible to sketch the global flows in all parameter regimes (with one exception because of the limitation in Proposition 12) in a systematic way by superimposing them on figures that include the lines defined by

$$-v_1 - C_{11}\rho_1 - C_{12}\rho_2 = 0 \tag{69}$$

$$-v_2 - C_{21}\rho_1 - C_{22}\rho_2 = 0, \tag{70}$$

on which $\dot{\rho}_1 = 0$ and $\dot{\rho}_2 = 0$, respectively. These lines, of course, were used extensively in establishing the ω-limit sets.

Figures 12–15 depict a complete coverage of the physical parameter space when $C_{11} \neq 0$ and $C_{22} \neq 0$ in the form of two-dimensional plots. Not every possible combination of parameters is presented, because the form of the equations allows a permutation of the indices. Thus, while the case $v_1 \geqslant 0$, $v_2 < 0$, $C_{12} \geqslant 0$, and $C_{21} < 0$ is not in the figures, it is analogous to the case $v_1 < 0$, $v_2 \geqslant 0$, $C_{12} < 0$, and $C_{12} \geqslant 0$ presented in Fig. 14. Each plot shows the parameter space broken into regions in which the flows are qualitatively equivalent. The boundaries of these regions (heavy lines) correspond to transcritical bifurcation points. Within each region a sketch of the flow in Q is presented along with the construction lines, Eqs. (69) and (70). The proposition or corollary to which each region corresponds is indicated by PN or CN. As a boundary is crossed, a transcritical bifurcation occurs and the flow shown on one side changes to the flow shown on the other side.

In four of the regions, which are depicted and marked by asterisks, in which there is a fixed point in the interior of Q and $C_{12}C_{21} < 0$, the flows may be misleading. Proposition 12 did not rule out the existence of one or

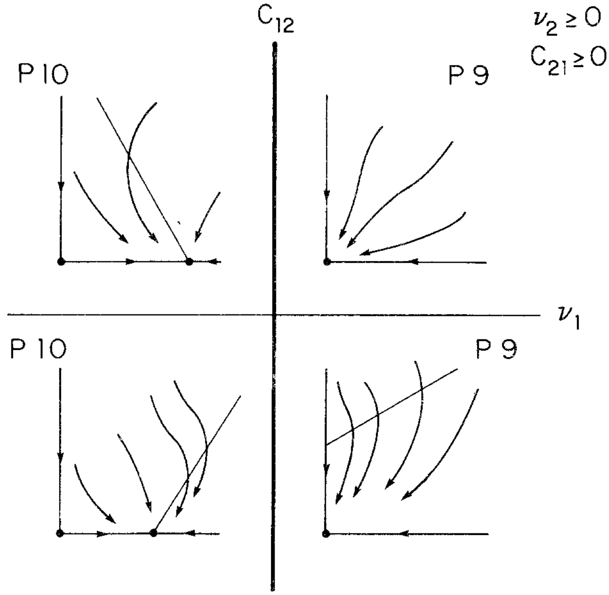

Fig. 12. Bifurcation points (heavy lines) for $\nu_2 \geqslant 0$, $C_{21} \geqslant 0$. In each region of parameter space the typical flow is sketched.

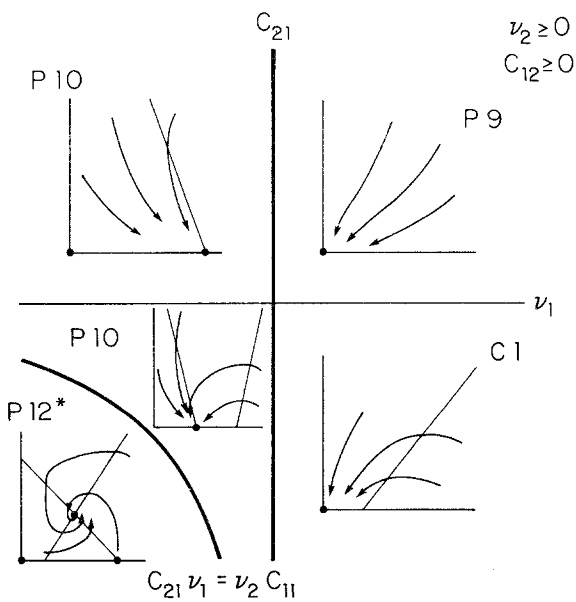

Fig. 13. Bifurcation points (heavy lines) for $\nu_2 \geqslant 0$, $C_{12} \geqslant 0$. The equation of the curve in the third quadrant is $C_{21}\nu_1 = \nu_2 C_{11}$.

more closed orbits in Q. Equation (64) did provide a criterion for their nonexistence, but for fixed values of C_{11} and C_{22} it is always possible to choose $(C_{12}C_{21})^2$ large enough to violate that criterion. This would not then imply the existence of a periodic orbit, but it would leave the question open. As discussed previously, to settle the matter it is only necessary to study the unstable manifold of $(-\nu_1/C_{11}, 0)$ [or indeed $(0, -\nu_2/C_{22})$]. Numerically obtained approximations to this manifold are presented in Fig. 16. They indicate that if a periodic orbit were to exist for the case studied, its amplitude would have to be smaller than the resolution of the numerical calculation. It seems unlikely that any such closed orbits exist in

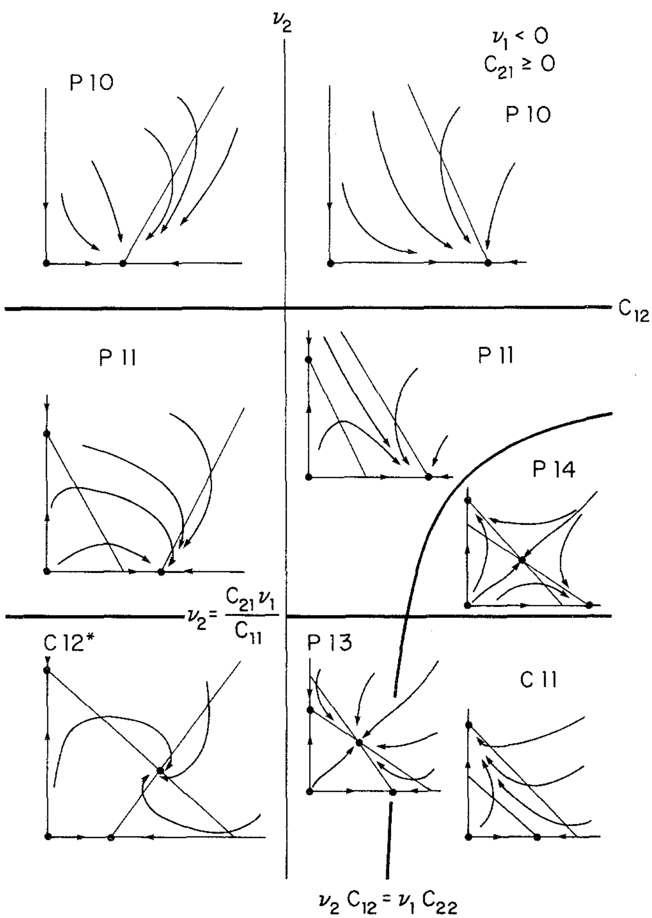

Fig. 14. Bifurcation points (heavy lines) for $\nu_1 < 0$, $C_{21} \geq 0$. The equation of the curve in the fourth quadrant is $\nu_2 C_{12} = \nu_1 C_{22}$.

this parameter regime. No bifurcations occur at any of the fixed points; so a periodic orbit would have to be born in some kind of global bifurcation when $(C_{12}C_{21})^2 > (C_{11}C_{22})^2$.

Considering the restrictions on interaction processes used to derive the model equations (3), it is important to know how the introduction of small additional processes (e.g., the production of species i in a j–k interaction) affects the evolution of the system. For the two-dimensional case a theorem from the dynamical systems literature, Peixoto's theorem,[5] and Proposition 7 allow a complete understanding of this question. The theorem only applies to systems on compact sets, hence the invocation of Proposition 7. For systems on a compact subset of $\mathbb{R}^2$ the theorem asserts

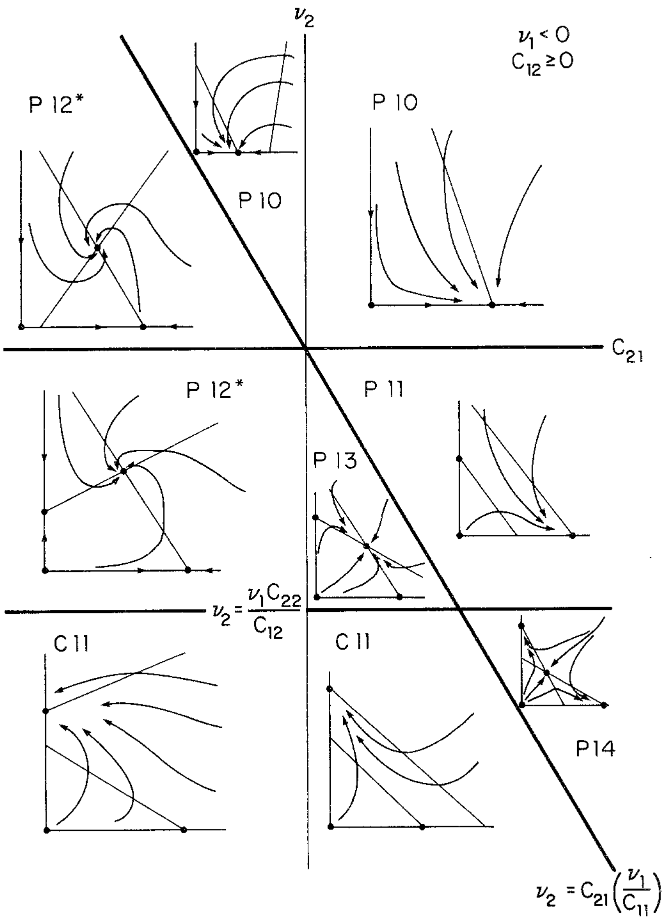

Fig. 15. Bifurcation points (heavy lines) for $\nu_1 < 0$, $C_{12} \geq 0$.

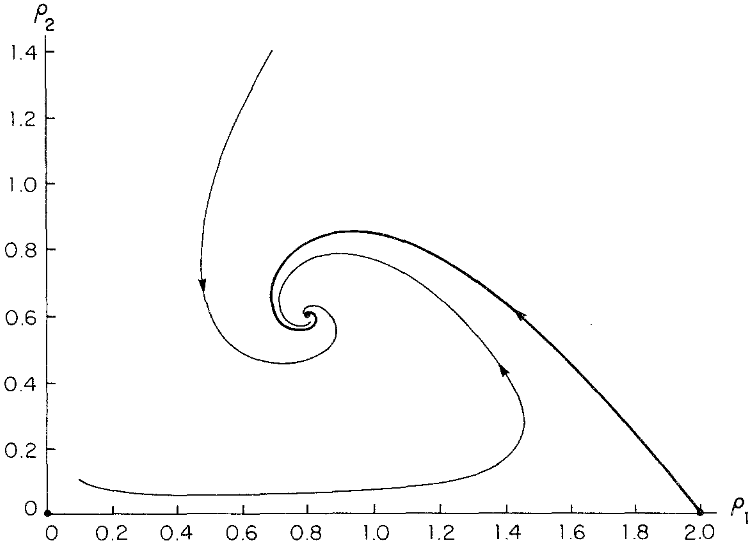

Fig. 16. Numerically computed trajectories for $\nu_1 = -2$, $\nu_2 = 1$, $C_{11} = 1$, $C_{12} = 2$, $C_{21} = -2$, and $C_{22} = 1$. The heavy curve is the unstable manifold of the fixed point at $(2, 0)$. Note that $(C_{12}C_{21}/C_{11}C_{22})^2 = 16 > 1$.

that if there are only a finite number of fixed points at each of which the stability-determining eigenvalues have nonzero real part, if there are no periodic orbits, and if there are no orbits connecting saddle points, then the system is structurally stable. The two-species equations (36) satisfy these conditions for almost all parameter values; only the bifurcation values are definitely excluded. This means that the addition of any small (and physical) terms on the right side of Eqs. (36) will not have any qualitative (and only a small quantitative) effect on the typical behavior of the system. Thus, if the interaction processes that were excluded to derive Eqs. (36) are in fact small, they can be neglected without introducing any qualitative effect. If there are in fact periodic orbits in some parameter regimes (Proposition 12), then Peixoto's theorem ensures that the system is structually stable if there are only a finite number of periodic orbits and if none of the stability-determining Floquet multipliers is equal to one in modulus.

Some general conclusions can be drawn concerning the types of behavior that Eqs. (36), and the gas mixture they model, can display provided $C_{11} \neq 0$, $C_{22} \neq 0$. In Table III the various types of behavior are summarized. If the periodic orbit associated with $\bar{\rho}_1 > 0$, $\bar{\rho}_2 > 0$, $C_{12}C_{21} < 0$ does not exist, as the numerical results suggest, then the presence of any self-removal ($C_{11} \neq 0$, $C_{22} \neq 0$) at all requires the system to approach a steady state. In general any initial preparation of the gas mixture, provided

Table III. Types of Behavior Exhibited by Two-Species Gas Mixture Model (for $C_{11} > 0$, $C_{22} > 0$)[a]

Parameter range	Behavior
$\nu_1 \geqslant 0$, $\nu_2 \geqslant 0$, $\bar{\rho}_1 \leqslant 0$, and/or $\bar{\rho}_2 \leqslant 0$	Both species die away to zero number density
$\nu_1 < 0$, $\nu_2 \geqslant 0$ or $\nu_2 < 0$, $\nu_1 \geqslant 0$, $\bar{\rho}_1 \leqslant 0$ and/or $\bar{\rho}_2 \leqslant 0$	One species dies away and the system achieves equilibrium with one species surviving
$\bar{\rho}_1 > 0$, $\bar{\rho}_2 > 0$, $C_{12}C_{21} < 0$ (which implies $\nu_1 < 0$ or $\nu_2 < 0$ or $\nu_1 < 0$ and $\nu_2 < 0$)	Both species survive and approach either an equilibrium or a periodic cycle (Fig. 16)
$\bar{\rho}_1 > 0$, $\bar{\rho}_2 > 0$, $C_{12}C_{21} > 0$, $C_{11}C_{22} - C_{12}C_{21} > 0$ (which implies $\nu_1 < 0$ and $\nu_2 < 0$)	Both species survive and approach an equilibrium (Fig. 17)
$\bar{\rho}_1 > 0$, $\bar{\rho}_2 > 0$, $C_{11}C_{22} - C_{12}C_{21} < 0$ (which implies $\nu_1 < 0$ and $\nu_2 < 0$	Only one species survives and approaches an equilibrium; which species survives depends upon the initial conditions (Fig. 18)

[a] $\bar{\rho}_1 = (C_{21}\nu_1 - C_{11}\nu_2)/(C_{11}C_{22} - C_{12}C_{21})$ and $\bar{\rho}_2 = (C_{12}\nu_2 - C_{22}\nu_1)/(C_{11}C_{22} - C_{12}C_{21})$.

both species are actually present, will evolve to the same steady state (cf. Fig. 17). For one parameter range, however, the system shows great sensitivity to its initial preparation. When $\bar{\rho}_1 > 0$, $\bar{\rho}_2 > 0$, and $C_{11}C_{22} - C_{12}C_{21} < 0$ the system can evolve to either of the steady states $(-\nu_1/C_{11}, 0)$ and $(0, -\nu_1/C_{22})$, depending on the initial number densities (cf. Fig. 18). Of course, it also could evolve to $(\bar{\rho}_1, \bar{\rho}_2)$, but this requires very special and precise initial conditions exactly on the stable manifold of $(\bar{\rho}_1, \bar{\rho}_2)$. Such initial conditions are unlikely to be achieved, since in attempting to produce such a special initial condition, tiny, random uncertainties will result, so that they are not on this manifold; hence, they will be on one side or the other, resulting in an apparently random selection of which species is to survive and which equilibrium is to be approached.

The global flow structure sketched here can be compared to explicit solutions of the two-species equations obtained by Boffi and Spiga[(10)] for special cases of the parameter values. The asymptotic behavior exhibited by their special solutions agrees with the general results obtained here.

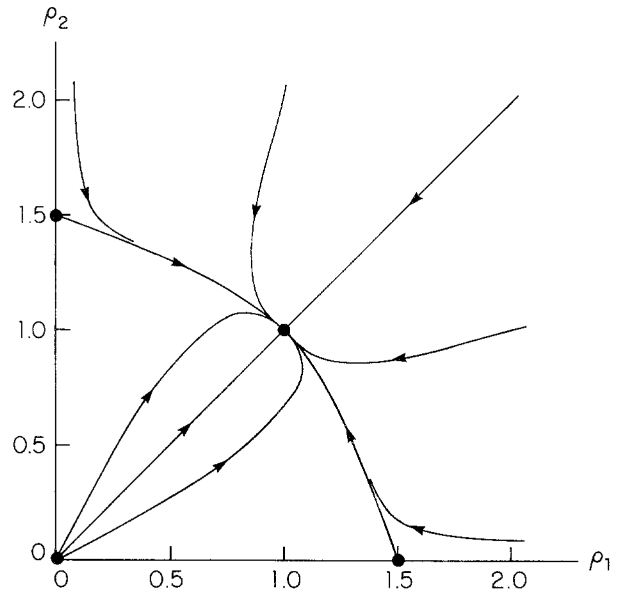

Fig. 17. Numerically computed trajectories for $v_1 = -3$, $v_2 = -3$, $C_{11} = 2$, $C_{12} = 1$, $C_{21} = 1$, and $C_{22} = 2$. All trajectories approach the interior fixed point as required by Proposition 13.

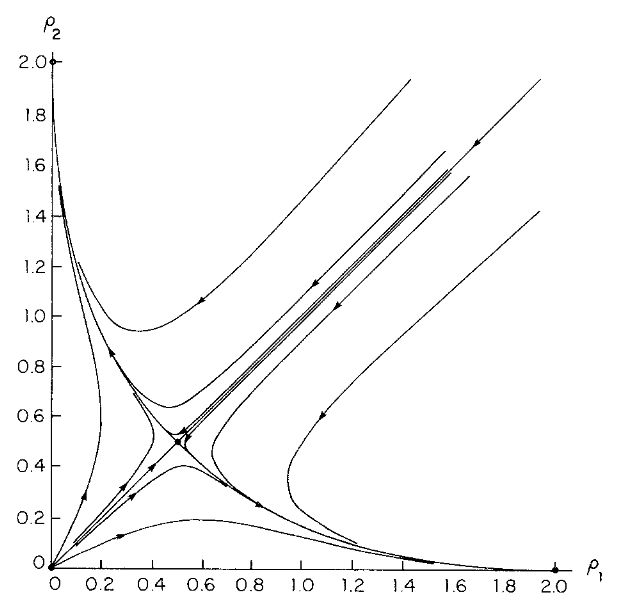

Fig. 18. Numerically computed trajectories for $v_1 = -2$, $v_2 = -2$, $C_{11} = 1$, $C_{12} = 3$, $C_{21} = 3$, and $C_{22} = 1$. The separatrix (45° line) clearly separates the basins of attraction of the two fixed points (2, 0) and (0, 2).

3.5. Two-Species Systems Without Self-Removal

The previous analysis of the two-species equations has consistently been based on the conditions $C_{11} > 0$ and $C_{22} > 0$ throughout. These conditions require that the collision of two identical gas molecules sometimes results in the destruction or loss of one or both molecules. Of course, a case with $C_{11} < 0$ or $C_{22} < 0$ is not physical and of no concern. But cases with $C_{11} = 0$ and/or $C_{22} = 0$ are physically relevant and must be considered.

If there is no self-removal, then the previous analysis is no longer valid. In particular, the proof of Proposition 7 no longer is adequate and a compact, positive invariant set may not exist. There is certainly not a positive invariant box like K for all parameter values: suppose $C_{11} = 0$ and $\nu_1 < 0$; then the flow on the invariant ρ_1 axis is governed by $\dot{\rho}_1 = -\nu_1 \rho_1 > 0$ and the density of species 1 grows without bound. Analogously, if $C_{22} = 0$ and $\nu_2 < 0$, species 2 grows without bound. Because of the limited interaction processes allowed in the model, a single gas without self-removal can only be consumed or produced by the background gas.

Not all of the lemmas and corollaries associated with Proposition 7 are lost, however. In particular, if $C_{11} = 0$ but $C_{22} > 0$, then Lemmas 1 and 3 are easily modified. These lemmas (and the corollary to Lemma 3) can then be used to establish $(0, 0)$ as the ω-limit of every point in Q for the associated parameter regimes ($C_{11} = 0$, $C_{12} > 0$, $C_{22} > 0$, $\nu_1 > 0$, $\nu_2 > 0$) (Fig. 19). It also is straightforward to extend the analysis by constructing a compact, positive invariant set for the range of parameters $C_{11} = 0$, $C_{12} > 0$, $C_{22} > 0$, $\nu_1 > 0$, $\nu_2 < 0$ and thereby show that $(0, -\nu_2/C_{22})$ is the ω-limit of every point not on the ρ_1 axis (Fig. 19).

When $C_{12} > 0$ and $C_{11} = 0$ the system becomes much more difficult to analyze, because it has not been possible to find compact, positive

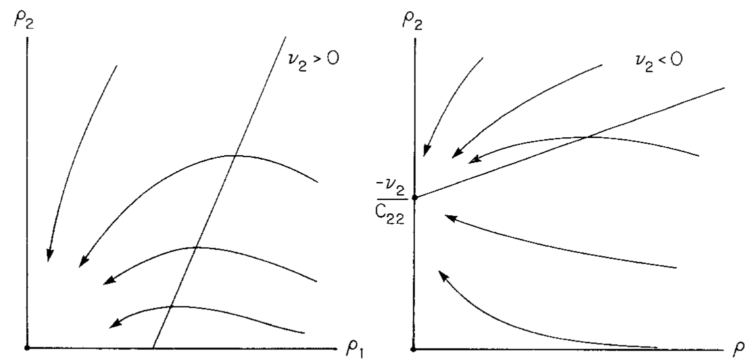

Fig. 19. Typical flows for the case $C_{11} = 0$ with $\nu_1 > 0$, $C_{22} > 0$, $C_{12} > 0$.

invariant sets containing arbitrary points of Q. In particular, it has not been possible to show that some solutions do not become infinite in finite time. For example, it has not been possible to show that pathological cases such as the one sketched in Fig. 20 do not occur.

One interesting case that can be fully analyzed is characterized by the conditions $C_{11} = C_{22} = 0$, $\nu_1 \nu_2 < 0$, $\nu_1 C_{12} < 0$, and $\nu_2 C_{21} < 0$. The equations are then the Volterra–Lotka equations of population biology. These equations have been studied extensively,[2] and are well understood. The flows consist of a family of concentric closed orbits in Q surrounding the stable fixed point $(-\nu_2/C_{21}, -\nu_1/C_{12})$ with amplitudes that depend upon the initial conditions (Fig. 21). To establish this it is necessary to construct a Liapunov function for the system; this is done in Ref. 2. As noted by Boffi *et al.*,[1] this flow can be thought of intuitively as occurring in a degenerate Hopf bifurcation as C_{11} and C_{22} go to zero.

3.6. A Manifold of Fixed Points

There may be some interest in the behavior of the system when a manifold of fixed points exists. An analysis of such a case will be presented here for the sake of completeness.

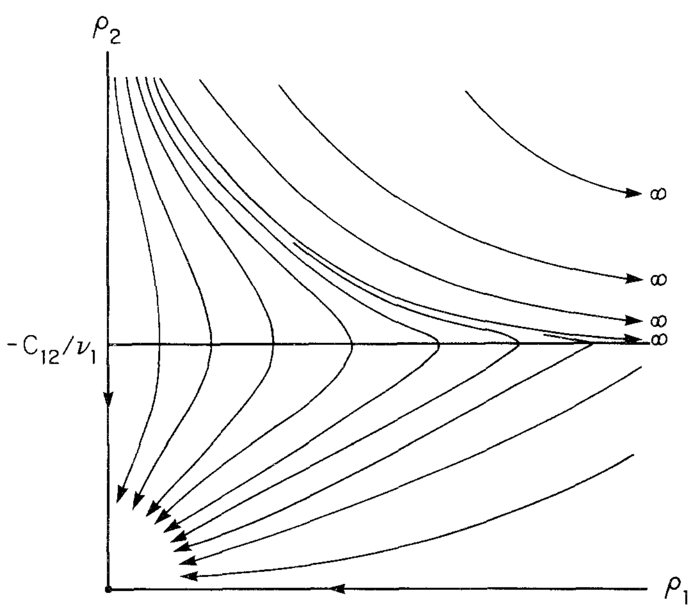

Fig. 20. A pathological flow of a type that might occur for $C_{11} = 0$, $C_{12} < 0$, $C_{21} > 0$, $\nu_1 > 0$, $\nu_2 > 0$.

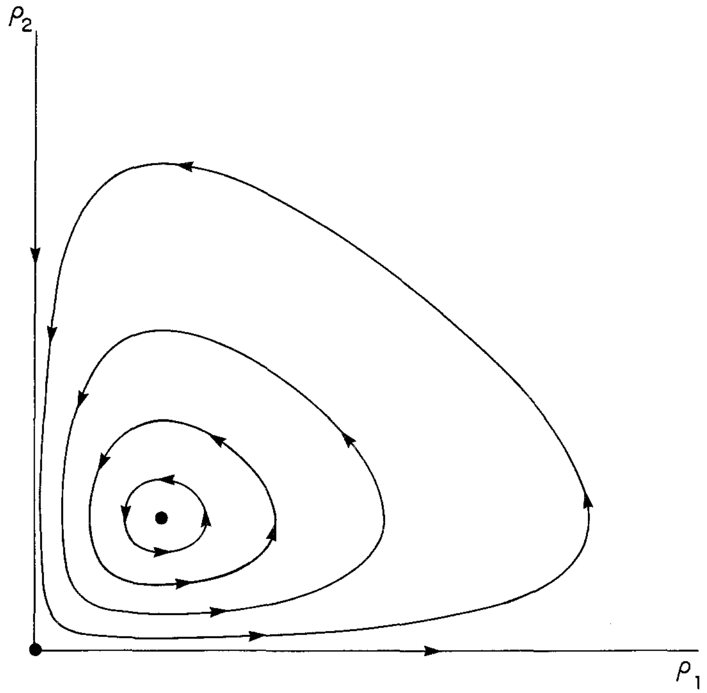

Fig. 21. The type of flow that occurs for $C_{11}=C_{22}=0$, $C_{12}>0$, $C_{21}<0$, $v_1<0$, $v_2>0$.

For the parameter range $v_1 C_{21} - v_2 C_{11} = 0$, $v_2 C_{12} - v_1 C_{22} = 0$, $v_1 < 0$, $v_2 < 0$, $C_{11} > 0$, and $C_{22} > 0$ there is a line of fixed points in Q extending from $(-v_1/C_{11}, 0)$ to $(0, -v_1/C_{22})$.

This manifold of fixed points arises in a bifurcation from three fixed points, $(-v_1/C_{11}, 0)$, $(\bar{\rho}_1, \bar{\rho}_2)$, and $(0, -v_2/C_{22})$; as the parameters are further varied, the manifold immediately bifurcates back to the three fixed points, which as a result have undergone changes of stability. This bifurcation corresponds schematically to the transition in Fig. 15 from P14 to P13 by passing through the intersection of the bifurcation sets $v_2 = C_{21}(v_1/C_{11})$ and $v_2 = C_{22}(v_1/C_{12})$.

Using Proposition 4 and Eq. (15), it is immediate that the eigenvalues of the Jacobian at any point $(\tilde{\rho}_1, \tilde{\rho}_2)$ on the manifold are

$$\lambda_1 = 0 \tag{71}$$

$$\lambda_2 = -\tilde{\rho}_1 C_{11} - \tilde{\rho}_2 C_{22} < 0 \tag{72}$$

It is easily verified that the appropriate eigenvectors are $(-C_{12}/C_{11}, 1)$ and $(\tilde{\rho}_1, (C_{22}/C_{12})\,\tilde{\rho}_2)$, respectively. The vector $(-C_{12}/C_{11}, 1)$ of the zero eigen-

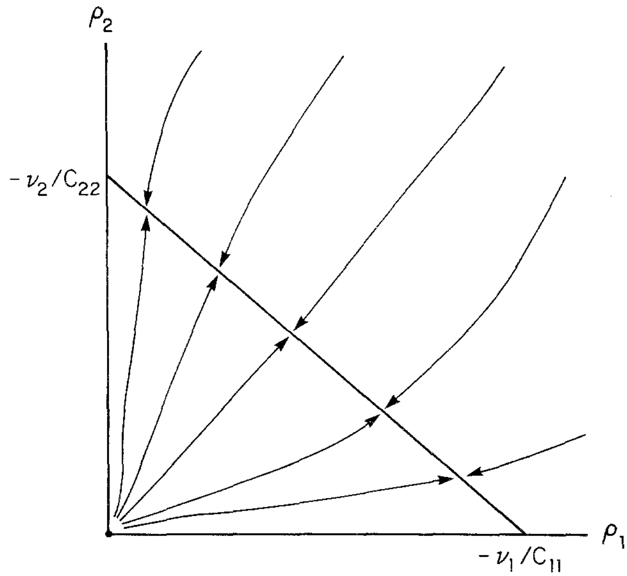

Fig. 22. An attracting manifold of fixed points.

value is tangent to the manifold of fixed points, and this manifold is therefore a center manifold for each of the fixed points in it. Since $\lambda_2 < 0$, it follows that this manifold is locally, exponentally attracting (this result is a direct consequence of Palmer's linearization theorem[11] or the center manifold theorem[8]).

Now, given any point $p \in Q$, $p \neq (0, 0)$, there is a compact, positive invariant box K (Proposition 7) containing $(0, 0)$, p, and every $(\tilde{\rho}_1, \tilde{\rho}_2)$. Thus, $\omega(p) \neq \varnothing$. If $\omega(p)$ does not intersect the manifold of fixed points, then the Poincaré–Bendixson theorem applies and $\omega(p) = (0, 0)$, but this will produce a contradiction as in previous proofs. However, if $q \in \omega(p)$ and $q = (\tilde{\rho}_1, \tilde{\rho}_2)$ for some $(\tilde{\rho}_1, \tilde{\rho}_2)$ on the manifold, then $\omega(p) = q$, since Palmer's theorem[11] provides a local stable manifold for each fixed point.

Thus, each initial state [except $(0, 0)$] is asymptotivally attracted to a single point on the manifold of fixed points (Fig. 22).

4. RECONSTRUCTION OF DISTRIBUTION FUNCTIONS

Naturally, the distribution functions $f_i(\mathbf{v}, t)$ for the species that comprise a gas are of interest. Hence, it is desirable to know how properties of the flow generated by Eqs. (3) on Q, $(\bar{\mathbb{R}}^+)^N$, are reflected in the properties of the flow generated by Eq. (1) on the phase space of almost everywhere non-negative integrable functions $(\bar{L}_1{}^+)^N$. In this section the connection between these two systems will be explicitly developed for the special case in which there is no external force field and the kernels depend only upon the molecular velocities with which the ith-species particles leave the collision events, i.e.,

$$F_i = 0 \tag{73}$$

$$g_{ij}^s \Pi_{ij}(\mathbf{v}', \mathbf{w}' \to \mathbf{v}) = g_{ij}^s \Pi_{ij}(\mathbf{v}) \tag{74}$$

$$g_{ij,i}^c \chi_{ij}(\mathbf{v}', \mathbf{w}' \to \mathbf{v}) = g_{ij,i}^c \chi_{ij}(\mathbf{v}) \tag{75}$$

This special class of scattering kernels preserves probabilities and therefore conserves species number densities, but not momentum or energy. For this special case, Eqs. (1) become

$$\frac{\partial f_i}{\partial t}(\mathbf{v}, t) = \sum_{j=1}^{N+1} \{-g_{ij} f_i \rho_j + V_{ij}(\mathbf{v})\, \rho_i \rho_j\} \tag{76}$$

where

$$V_{ij}(\mathbf{v}) = g_{ij}^s \Pi_{ij}(\mathbf{v}) + g_{ij,i}^c \chi_{ij}(\mathbf{v}) \tag{77}$$

and the integrals over $\mathbf{w}'$ and $\mathbf{v}'$ have been performed. These equations generate a flow on $(\bar{L}_1^+)^N$ with fixed points $(\bar{f}_1 \cdots \bar{f}_N) \in (L_1)^N$ determined by

$$0 = \sum_{j=1}^{N+1} \{-g_{ij} \bar{f}_i \tilde{\rho}_j + V_{ij}(\mathbf{v})\, \tilde{\rho}_i \tilde{\rho}_j\}, \qquad i = 1, 2,..., N \tag{78}$$

where $\tilde{\rho}_j = \int \bar{f}_j(v)\, dv$ is simply some number. Solving for $\bar{f}_i$ for every i gives

$$\bar{f}_i(\mathbf{v}) = \tilde{\rho}_i \frac{\sum_{j=1}^{N+1} V_{ij}(\mathbf{v})\, \tilde{\rho}_j}{\sum_{j=1}^{N+1} g_{ij} \tilde{\rho}_j} \tag{79}$$

Integrating over $\mathbf{v}$ gives an equation that the $\tilde{\rho}_j$ must satisfy for $\bar{f}_i$, $i = 1, 2,..., N$, to be a fixed point. This equation is

$$0 = \tilde{\rho}_i \sum_{j=1}^{N+1} [g_{ij}^R + g_{ij,j}^c + g_{ij,i}^c(1 - \eta_{ij})]\, \tilde{\rho}_j \tag{80}$$

or, using the definitions of C_{ij} and v_i,[1]

$$0 = \tilde{\rho}_i \left(-v_i - \sum_{j=1}^{N} C_{ij} \tilde{\rho}_j \right) \tag{81}$$

These are just the fixed point equations (6) for the number densities. Thus, the $\tilde{\rho}_i$ are identical to the $\bar{\rho}_i$, and Eqs. (79) and (81) provide a one-to-one correspondence between the fixed points for Eqs. (3) and (1) for the special case [Eqs. (73)–(75)] under consideration.

If Eqs. (3) are solved for $\rho_i(t)$, $i=1, 2,..., N$, then Eqs. (76) can be solved easily for $f_i(\mathbf{v}, t)$ as

$$\begin{aligned} f_i(\mathbf{v}, t) = f_i(\mathbf{v}, 0) \exp\left[-\sum_{j=1}^{N+1} g_{ij} \int_0^t \rho_j(\tau)\, d\tau\right] \\ + \sum_{j=1}^{N+1} V_{ij}(\mathbf{v}) \int_0^t \rho_i(t')\, \rho_j(t') \\ \times \exp\left[-\sum_{j=1}^{N+1} g_{ij} \int_{t'}^t \rho_j(\tau)\, d\tau\right] dt' \end{aligned} \tag{82}$$

As a first case, consider any of the infinite number of initial distribution functions that satisfy

$$\bar{\rho}_i = \int f_i(\mathbf{v}, 0)\, d\mathbf{v} \tag{83}$$

for some fixed point $(\bar{\rho}_1,..., \bar{\rho}_N)$. Then, $\int f_i(v, t)\, dv = \bar{\rho}_i$ for all time and

$$\begin{aligned} f_i(\mathbf{v}, t) = f_i(\mathbf{v}, 0) \exp\left(-\sum_{j=1}^{N+1} g_{ij}\bar{\rho}_j t\right) \\ + \frac{\sum_{j=1}^{N+1} V_{ij}(\mathbf{v})\, \bar{\rho}_i \bar{\rho}_j}{\sum_{j=1}^{N+1} g_{ij}\bar{\rho}_j} \left[1 - \exp\left(-\sum_{j=1}^{N+1} g_{ij}\bar{\rho}_j t\right)\right] \end{aligned} \tag{84}$$

Since $g_{ij} \geqslant 0$ and $\bar{\rho}_j \geqslant 0$, it is obvious that, provided $\sum_{j=1}^{N+1} g_{ij}\bar{\rho}_j \neq 0$,

$$\lim_{t\to\infty} f_i(\mathbf{v}, t) = \frac{\sum_{j=1}^{N+1} V_{ij}(\mathbf{v})\, \bar{\rho}_i\bar{\rho}_j}{\sum_{j=1}^{N+1} g_{ij}\bar{\rho}_j}, \qquad i=1, 2,..., N \tag{85}$$

which is just the fixed point in $(\bar{L}_1^+)^N$ associated with $(\bar{\rho}_1 \cdots \bar{\rho}_N)$ in Q. Thus, the system (76) decays to the fixed point, or equilibrium velocity distribution, exponentially with rate $\sum_{j=1}^{N+1} g_{ij}\bar{\rho}_j$, which is the total interaction frequency.

More generally, consider any initial distribution $f_i(\mathbf{v}, 0)$, $i=1, 2,..., N$, with corresponding number densities $\rho_i(0)$, $i=1, 2,..., N$. Suppose that $\rho_i(t) \to \bar{\rho}_i$ exponentially for all $i=1, 2,..., N$, where $(\bar{\rho}_1,..., \bar{\rho}_N)$ is some fixed point of Eqs. (3) in Q, i.e., $\{\bar{\rho}\}$ is the omega-limit set of $\rho(0)$ and

$$\|\bar{\rho} - \rho(t)\| \leqslant \begin{cases} M, & t \leqslant T \\ Me^{-\sigma t}, & t > T \end{cases} \tag{86}$$

for some M, σ, $T>0$. This requires that if $g_{ij}\rho_j \to 0$ for every $j<N+1$ as $t\to\infty$, then $g_{iN+1}\rho_{N+1} \neq 0$ (recall that species $N+1$ is the background gas). Note also that if $g_{ij}\rho_j \to g_{ij}\bar{\rho}_j \neq 0$, then $g_{ij}\rho_j$ is bounded below by a strictly positive number by the invariance properties discussed in Proposition 1.

Writing $\eta_i(t)=\bar{\rho}_i-\rho_i(t)$, we find the exact solution for $f_i(\mathbf{v}, t)$ as

$$\begin{aligned} f_i(\mathbf{v}, t) ={}& f_i(\mathbf{v}, 0)\exp\left(-\sum_{j=1}^{N+1} g_{ij}\bar{\rho}_j t\right)\exp\left[-\sum_{j=1}^{N+1} g_{ij}\int_0^t \eta_j(\tau)\,d\tau\right] \\ &+\sum_{j=1}^{N+1} V_{ij}(\mathbf{v})\int_0^t \bar{\rho}_i\bar{\rho}_j \exp\left[-\sum_{j=1}^{N+1} g_{ij}\bar{\rho}_j(t-t')\right] \\ &\times\exp\left[-\sum_{j=1}^{N+1} g_{ij}\int_{t'}^t \eta_j(\tau)\,d\tau\right]dt' \\ &+\sum_{j=1}^{N+1} V_{ij}(\mathbf{v})\int_0^t (\bar{\rho}_i\eta_j+\bar{\rho}_j\eta_i+\eta_i\eta_j) \\ &\times\exp\left[-\sum_{j=1}^{N+1} g_{ij}\int_{t'}^t \rho_j(\tau)\,d\tau\right]dt', \qquad i=1, 2,..., N \end{aligned} \tag{87}$$

Consider an integral of the form

$$I=\int_0^t \eta_i(t')\exp\left[-\sum_{j=1}^{N+1} g_{ij}\int_{t'}^t \rho_j(\tau)\,d\tau\right]dt' \tag{88}$$

If $g_{ij}\rho_j \to 0$ for every $j<N+1$, then

$$\exp\left[-\sum_{j=1}^{N+1} g_{ij}\int_{t'}^t \rho_i(\tau)\,d\tau\right] \leqslant \exp[-g_{iN+1}\rho_{N+1}(t-t')] \tag{89}$$

since $\rho_i(t)\geqslant 0$ for all t, and if $g_{ij}\rho_j \nrightarrow 0$, then

$$\exp\left[-\sum_{j=1}^{N+1} g_{ij}\int_{t'}^t \rho_j(\tau)\,d\tau\right] \leqslant \exp[-\alpha(t-t')] \tag{90}$$

for some $\alpha>0$. For either case, using the exponential bound on η_i, we obtain an estimate of the form

$$|I| \leqslant \int_0^T Me^{-K(t-t')}\,dt' + \int_T^t Me^{-\sigma t'}e^{-K(t-t')}\,dt' \tag{91}$$

(with $K=g_{iN+1}\rho_{N+1}$ or $K=\alpha$). So

$$|I| \leqslant Me^{-Kt}\left[\frac{1}{K}e^{KT}-\frac{1}{K}-\frac{e^{(K-\sigma)T}}{K-\sigma}\right]+Me^{-\sigma t}\left[\frac{1}{K-\sigma}\right] \tag{92}$$

and $I \to 0$ as $t \to \infty$. Since $\eta_i(t)\,\eta_j(t)$ also goes to zero exponentially, the integral

$$\int_0^t \eta_i \eta_j \exp\left[-\sum_{j=1}^{N+1} g_{ij} \int_{t'}^t \rho_j(\tau)\, d\tau \right] dt' \tag{93}$$

admits a similar estimate and goes to zero exponentially.

The exponential decay of η_j ensures that

$$\left| \int_0^\infty \eta_j(\tau)\, d\tau \right| < \infty \tag{94}$$

and so

$$f_i(\mathbf{v}, 0) \exp\left(-\sum_{j=1}^{N+1} g_{ij} \bar{\rho}_j t \right) \exp\left[-\sum_{j=1}^{N+1} g_{ij} \int_0^t \eta_j(\tau)\, d\tau \right] \tag{95}$$

also goes to zero as $t \to \infty$. Only the terms

$$\begin{aligned} J = \sum_{j=1}^{N+1} V_{ij}(\mathbf{v}) \int_0^t \bar{\rho}_i \bar{\rho}_j \exp\left[-\sum_{j=1}^{N+1} g_{ij} \bar{\rho}_j (t - t') \right] \\ \times \exp\left[\sum_{j=1}^{N+1} g_{ij} \int_{t'}^t \eta_j(\tau)\, d\tau \right] dt' \end{aligned} \tag{96}$$

remain. Integrating by parts yields

$$\begin{aligned} J = \frac{\sum_{j=1}^{N+1} V_{ij}(\mathbf{v})\, \bar{\rho}_i \bar{\rho}_j}{\sum_{j=1}^{N+1} g_{ij} \bar{\rho}_j} \Bigg(\Bigg\{ 1 - \exp\left(-\sum_{j=1}^{N+1} g_{ij} \bar{\rho}_j t \right) \\ \times \exp\left[-\sum_{j=1}^{N+1} g_{ij} \int_0^t \eta_j(\tau)\, d\tau \right] \Bigg\} \\ - \sum_{k=1}^{N+1} g_{ik} \int_0^t \eta_k(t') \exp\left[-\sum_{j=1}^{N+1} g_{ij} \int_{t'}^t \rho_j(\tau)\, d\tau \right] dt' \Bigg) \end{aligned} \tag{97}$$

In view of the previous estimates,

$$J \to \frac{\sum_{j=1}^{N+1} V_{ij}(\mathbf{v})\, \bar{\rho}_i \bar{\rho}_j}{\sum_{j=1}^{N+1} g_{ij} \bar{\rho}_j} \tag{98}$$

exponentially as $t \to \infty$. Thus, as $t \to \infty$, $f_i(v, t)$, $i = 1, 2, \ldots, N$, evolves to the fixed point $(\bar{f}_1(v), \ldots, \bar{f}_N(v))$ associated with $(\bar{\rho}_1, \ldots, \bar{\rho}_N)$.

This correspondence between exponential decay in the problem for the number densities and the problem for the distribution functions allows the geometry of the flows to be related. It is clear that to the stable manifold of a fixed point $\bar{\rho}$ there corresponds an inward-flowing set (in-set) of the corresponding fixed point $\bar{f}$. It is not clear that this in-set for $\bar{f}$ is a manifold, but if it is, then it must be infinite-dimensional. Further, since the map from $(\bar{L}_1^+)^N$ to $(\bar{\mathbb{R}}^+)^N$ defined by integration over all velocity is continuous (but note that it does not have an inverse), if $\bar{\rho}$ is a stable hyperbolic (i.e., exponentially attracting) fixed point, then $\bar{f}$ is stable.

In this section degenerate kernels were used in the interaction integrals and the distribution function reconstructed from the number densities. In the scattering integrals the kernel used is one that can be made to preserve particle numbers, but not momentum and energy in a collision. The other interaction kernels were written in an analogous form. Within the context of this additional restriction on the model equations[1] studied here, and for the case of no external forces, a one-to-one correspondence exists between the equilibria for the number densities and the equilibria for the distribution functions. Furthermore, exponential stability of a number density equilibrium in Q implies the exponential stability of the corresponding distribution function equilibrum. However, it is not at all clear that this is a general result that might also be true in the case of more realistic kernels.

It is also possible to reconstruct the distribution function in the case with nonzero and time-dependent external forces (but still with the same degenerate kernels). This extension of the construction presented here was brought to the attention of the authors by V. Protopopescu. The results relating equilibria do not extend so easily, however; in fact, in the case of time-dependent external forces there are no equilibria for the coupled Boltzmann equations even when there are fixed points for the number density equations.

5. CONCLUSIONS

Boffi *et al.*[1] proposed Eqs. (1) as a theoretical model for the study of the nonlinear time evolution of a multispecies gas undergoing binary collision creation, scattering, and removal processes. In the present paper some of the mathematical consequences of the model have been studied in order to shed light on the time evolution of such gases.

For the full N-species equations (3) it was straightforward to show that the solutions remain positive for as long as they exist. This means that the model does not allow an unphysical dominance of the removal processes; as all the particles of some species are consumed, the removal mechanism shuts down in a physically reasonable way.

Particular emphasis was placed on the location of fixed points or equilibria where the creation and removal processes exactly balance. The special form of the equations allows the fixed-point problem to be completely understood. The static bifurcation behavior of the problem is thereby solved: the system may exhibit (possibly multiple) transcritical bifurcations and a bifurcation from isolated fixed points to manifolds of fixed points. It was possible to make statements concerning the local behavior of the system when it was near such equilibria, especially for parameters in the dense good subset of the full physical parameter set. Some general criteria for a fixed point to be unstable were developed. The existence of the contracting direction at a fixed point $(\bar{\rho}_1, ..., \bar{\rho}_N)$ provided that $\bar{\rho}_i C_{ii} \neq 0$ for some i is particularly interesting. At such an equilibrium there is a positive density of species i that undergoes self-removal, since $C_{ii} > 0$, but this removal process in fact ensures that there is some special set of initial conditions [stable manifold for $(\bar{\rho}_1, ..., \bar{\rho}_N)$] which evolve to the equilibrium. By exploiting a special relationship [Eq. (20)] between different fixed points, it was possible to show that fixed points enter Q via transcritical bifurcations at the boundary of Q. Based on this and the explicit stability criteria for the $(0, ..., 0)$ fixed point, it was shown that the full N-species model equations have stable equilibria, with any number from zero to N gas species surviving. The precise parameter ranges in which these fixed points exist were not identified, although necessary conditions for such a fixed point to have been created by a specific sequence of bifurcations were given.

In the two-species case ($N = 2$) the Poincaré–Bendixson theorem was used to establish the asymptotic behavior of every initial condition in Q when self-removal is present ($C_{11} > 0$, $C_{22} > 0$). For most parameter regimes almost all initial states evolve to a single equilibrium. The parameter range in which a separatrix splits Q into two sets, each the basin of attraction for a fixed point, also was determined. There is a possibility that in one parameter range the system might evolve to a periodic solution, but this is unlikely. The possibility could be excluded by examining the global unstable manifold of one fixed point.

Finally, it is important to recognize that the self-removal (C_{11}, C_{22}) and background creation/removal (v_1, v_2) terms are very important. There are qualitative differences in system behavior between cases with C_{ii} or v_i small and cases with $C_{ii} = 0$ or $v_i = 0$. Most importantly, neither species can survive unless $v_1 \neq 0$ and $v_2 \neq 0$; and for one species to survive it is necessary that $v_1 < 0$ or $v_2 < 0$.

ACKNOWLEDGMENT

This research was supported in part by the U.S. Office of Naval Research under contract N00014-85-K-0382.

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